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CasADi

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Outline

Motivation – Large-scale Optimal Control Problems (OCP)

$$
\begin{aligned}
&\min_{x,u,p} \int_0^T l(t,x,u,p)\,dt + E(x(T),p) \\
&\text{subj. to} \\
&\qquad \dot{x} = f(t,x,u,p) = 0 \quad t \in [0,T] \\
&\qquad h(t,x,u,p) \le 0 \quad t \in [0,T] \\
&\qquad x(0) = x_0 \\
&\qquad x_{\min} \le x \le x_{\max} \quad t \in [0,T] \\
&\qquad u_{\min} \le u \le u_{\max} \quad t \in [0,T] \\
&\qquad p_{\min} \le p \le p_{\max}
\end{aligned}
\tag{1}
$$

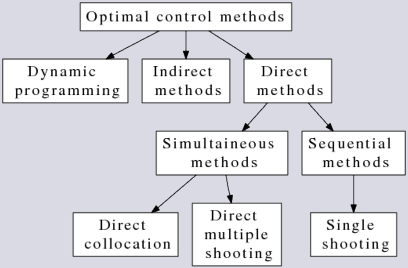
Here $x(\cdot) \in \mathbf{R}^{N_x}$ (differential) states, $u(\cdot) \in \mathbf{R}^{N_u}$ free control signals and $p \in \mathbf{R}^{N_p}$ free parameters.

Motivation

Solving optimal control problems

Methods for solving OCP's:

- *Dynamic programming/HJB*
  "Smart exhaustive search"
- *Indirect methods*
  "Solve necessary conditions for optimality"
- **Direct methods**
  "Reformulate as a nonlinear programming problem (NLP)"



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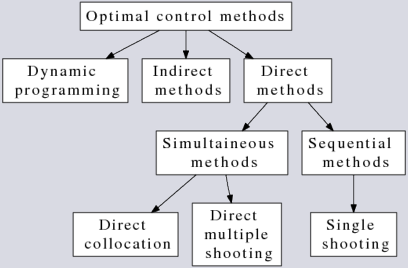

Direct methods

- Single shooting: parametrize only controls, eliminate state with ODE/DAE integrators
- Simultaneous methods: parametrize controls *and* state
  - Direct collocation: Fine grid – **interpolate** between gridpoints
  - Direct multiple shooting: Coarse grid – **integrate** between gridpoints
- Solve NLP with (structure exploiting) SQP or IP method

Automatic differentiation, AD

Efficient procedure to automatically calculate derivatives:

$$F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m \qquad \Rightarrow \qquad J(x) = \frac{\partial F}{\partial x}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n} \tag{2}$$

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How AD works

- Write F as a sequence of elementary operations:

$$\begin{aligned}
y_{i-n} &= x_i, & i &\in \{1, \ldots, n\} & &\text{independent inputs} \quad (3)\\
y_i &= f_i(y_{j_i}, y_{k_i}), \quad j_i < i,\ k_i < i & i &\in \{1, \ldots, p\} & &\text{intermediate calculations} \quad (4)\\
z_j &= y_{i_j}, & j &\in \{1, \ldots, m\} & &\text{function outputs} \quad (5)
\end{aligned}$$

- Consider the equality (with $y = [y_1, \ldots, y_p]$):

$$\begin{bmatrix} y \\ z \end{bmatrix} = \tilde{F}(x, y) \tag{6}$$

Now differentiate the extended equation

$$\begin{bmatrix} y \\ z \end{bmatrix} = \tilde{F}(x, y) \Rightarrow \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & L \\ B & M \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{7}$$

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  - From the left (*adjoint mode*): $v^\mathrm{T}\,J = v^\mathrm{T}\,B + v^\mathrm{T}\,M\,(I - L)^{-1}\,A$

Background

Now differentiate the extended equation

$$\left[\begin{array}{c} y \\ z \end{array}\right] = \tilde{F}(x, y) \Rightarrow \left[\begin{array}{c} \dot{y} \\ \dot{z} \end{array}\right] = \left[\begin{array}{cc} A & L \\ B & M \end{array}\right] \left[\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right] \tag{7}$$

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  - From the left (*adjoint mode*): $v^{\mathrm{T}}\,J = v^{\mathrm{T}}\,B + v^{\mathrm{T}}\,M\,(I - L)^{-1}\,A$
- To calculate the full Jacobian, multiply by several forward and/or adjoint directions
- NP-hard optimization problem to find the least number of directions
- AD software tools: ADOL-C, CppAD, OpenAD, ...

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Existing tools for OCP

- Tools exist that accept OCP:s in a standard form and solves the problem...
    - Shooting methods (e.g. MUSCOD-II, ACADO Toolkit)
    - Direct collocation (e.g. DIRCOL)
    - ...

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  - The user reformulates the OCP as an NLP
  - Derivative information is generated automatically and passed to the NLP solver
  - Advantages:
    - Can formulate arbitrarily complex non-standard OCP:s
    - User gets a better insight
  - Drawback:
    - Until now only for collocation methods

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`www.casadi.org`

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Automatic differentiation in CasADi

- CasADi applies AD to a sequence of vector/matrix valued operations
  - Conventional AD-tools support only unary or binary elementary operations (+, -, *, sin, sqrt, ...)
  - Matrix operations are *expanded* into a large set of scalar operations
- CasADi allows multiple matrix-valued input, multiple-matrix valued output: $[x, p] \rightarrow [f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial p}]$
- Also e.g. matrix multiplication, implicitly defined functions (solve linear or nonlinear system), etc.

How to get it to work efficiently?

- Matrix sparsity always fixed, sparsity shared between nodes
- Delay generation of Jacobians of function evaluations until the whole graph is known
- Use two different computational graph representations (scalar / matrix)
  - Avoids that the extra generality comes at the expense of memory use and speed

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Two graph representations

	Scalar-valued nodes	**Sparse, matrix-valued nodes**
design objective	"maximum speed"	"maximum generality"
type of operations	built-in	built-in or user-defined
number of inputs	one or two	arbitrary
number of output	one	arbitrary
branching/jumps	no	yes
parallelization	no	yes

"Smart interfaces" to numerical codes

- NLP solvers: **IPOPT**, **KNITRO**, **LiftOpt**

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- Other tools: **ACADO Toolkit**, **KINSOL**, **CPLEX**

Other features

- Just-in-time compilation
  - Generate C-code from CasADi computational graphs
  - Compile to a dynamic library
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- Import of DAE-systems formulated in Modelica
  - Modelica – DAE modelling language
- Symbolic reformulation of DAE:s
  - Sorting/scaling of variables and equations
  - Elimination of some or all algebraic states *symbolically*

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Direct collocation

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Shooting methods

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- Add ODE/DAE integrator calls to the computational graph
- Automatic formulation of ODE/DAE sensitivity equations
- Automatic ODE/DAE Jacobian information, preconditioning

Optimal control using CasADi

Code example: Complete single-shooting method in 30 lines of code!

```
from casadi import *

# Declare variables (use simple, efficient DAG)
t = SX("t") # time
x=SX("x"); y=SX("y"); u=SX("u"); L=SX("cost")

# ODE right hand side function
f = [(1 - y*y)*x - y + u, x, x*x + y*y + u*u]
rhs = SXFunction([[t],[x,y,L],[u]],[f])

# Create an integrator (CVodes)
I = CVodesIntegrator(rhs)
I.setOption("abstol",1e-10) # abs. tolerance
I.setOption("reltol",1e-10) # rel. tolerance
I.setOption("steps_per_checkpoint",100)
I.init()

# All controls (use complex, general DAG)
NU = 20; U = MX("U",NU)

# The initial state (x=0, y=1, L=0)
X  = MX([0,1,0])

# Time horizon
T0 = MX(0);  TF = MX(20.0/NU)

# State derivative, algebraic state (not used)
XP = MX(); Z = MX()

# Build up a graph of integrator calls
for k in range(NU):
  [X,XP,Z] = I.call([T0,TF,X,U[k],XP,Z])

# Objective function: L(T)
F = MXFunction([U],[X[2]])

# Terminal constraints: 0<=[x(T);y(T)]<=0
G = MXFunction([U],[X[0:2]])

# Create NLP solver
solver = IpoptSolver(F,G)
solver.setOption("tol",1e-5)
solver.setOption("hessian_approximation", \
                 "limited-memory")
solver.setOption("max_iter",1000)
solver.init()

# Set bounds and initial guess
solver.setInput(NU*[-0.75], NLP_LBX)
solver.setInput(NU*[1.0],NLP_UBX)
solver.setInput(NU*[0.0],NLP_X_INIT)
solver.setInput([0,0],NLP_LBG)
solver.setInput([0,0],NLP_UBG)

# Solve the problem
solver.solve()
```

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  - Automatic generation of derivative/sensitivity information
  - Automatic generation of sparse NLP Jacobians
  - A number of codes already written (multiple shooting/collocation,..)

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  - Bidirectional (combination of AD forward and adjoint)

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  - PDE constrained optimization?
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