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Nonlinear observer design using Lyapunov's auxiliary theorem

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Abstract

The present work proposes a new approach to the nonlinear observer design problem. Based on the early ideas that influenced the development of the linear Luenberger observer theory, the proposed approach develops a nonlinear analogue. The formulation of the observer design problem is realized via a system of singular first-order linear PDEs, and a rather general set of necessary and sufficient conditions for solvability is derived by using Lyapunov's auxiliary theorem. The solution to the above system of PDEs is locally analytic and this enables the development of a series solution method, that is easily programmable with the aid of a symbolic software package. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The availability of all the state variables for direct measurement is a rare occasion in practice. In most cases there is a true need for a reliable estimation of the unmeasurable state variables, especially when they are used for the synthesis of model-based controllers or for process monitoring purposes. For this particular task, a state observer is usually employed, in order to accurately reconstruct the state variables of the process. In the case of linear systems, the observer design theory developed by Luenberger [12], offers a complete and comprehensive answer to the problem. In the field of nonlinear systems, the nonlinear observer design problem is much more challenging and has received a considerable amount of attention in the literature. Numerous attempts have been made for the development of nonlinear observer design methods. One could mention the industrially popular extended Kalman filter, whose design is based on a local linearization of the system around a reference trajectory, restricting the validity of the approach within a small region in state space [7]. More recent attempts include Zeitz's extended Luenberger observer [17], which is in the same spirit as the extended Kalman filter, based upon a local linearization technique around the reconstructed state. In [1] a nonlinear observer design method is proposed, that places the eigenvalues of the linearized error equation at certain values, that are locally invariant with respect to the operating point of the system. The first systematic approach for the development of a theory of nonlinear observers was proposed by Krener and Isidori [11]. In their pioneering work, the authors made use of a nonlinear state transformation to linearize the original system up to an additional output injection term. Linear methods were then employed to complete the observer design procedure. However, this linearization approach is based upon a set of extremely restrictive conditions, that can hardly be met by any physical system. Nonlinear coordinate transformations have also been employed to

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transform the nonlinear system to a suitable "observer canonical form", where the observer design problem may be easily solved [2, 6, 14]. All these approaches however, impinge on the problem of the existence of certain sets of rather restrictive conditions in their theoretical body. Major comprehensive contributions to the nonlinear observer design problem can be found in recent pieces of research work, where a different type of approach is adopted, enabling the derivation of theoretically sound results [4, 8, 15, 16]. In particular, one should mention the high gain nonlinear observers proposed by Gauthier et al. [8], as well as the Luenberger-like nonlinear observers developed by Ciccarella et al. [4]. Both approaches however, rely on restrictive Lipshitz or Hölder continuity conditions. In addition, Tsinias [15, 16] proposed a novel approach to the above problem by establishing a Lyapunov-like sufficient condition for the existence of a nonlinear observer. However, the construction of an appropriate Lyapunov function-candidate remains still a difficult and challenging task.

Motivated by some preliminary results in the design of nonlinear observers that have been succesfully applied to the problem of catalyst activity estimation [9], the present work aims at the development of a systematic nonlinear observer design method. The proposed nonlinear observer design method generalizes Luenberger's early ideas on the problem [12], while at the same time it enjoys the advantage of being based on a fairly general set of conditions. In order to meet the main design objectives, the problem under consideration translates into the problem of solving a system of singular first-order linear partial differential equations (PDEs). The specific system of PDEs admits a unique and locally analytic solution, according to the so-called Lyapunov's auxiliary theorem [13]. Moreover, the analyticity of the solution of the above system of PDEs enables the development of a series solution method. Then, a nonlinear observer may be designed, possessing a nonlinear gain that is computed from the solution of the above system of PDEs.

2. Formulation of the nonlinear observer design problem

We consider single-output autonomous systems with the following state-space representation:

$$\dot{x} = f(x), \qquad y = h(x), \tag{1}$$

where $x \in R^n$ is the vector of state variables and $y \in R$ the output variable. It is assumed that $f(x)$ is a real analytic vector field on R^n and that $h(x)$ is a real analytic scalar field on R^n. Let the origin $x = 0$ be an equilibrium point of (1): $f(0) = 0$ with $h(0) = 0$. The following assumptions are made:

Assumption 1. The Jacobian matrix F of the $f(x)$ vector field evaluated at $x = 0$: $F = \partial f(0)/\partial x$ has eigenvalues $k_i, (i = 1, \dots, n)$ with

$$0 \notin CH\{k_1, k_2, \dots, k_n\}, \tag{2}$$

where CH stands for the convex hull of a set.

This assumption essentially implies that system (1) can be either locally asymptotically stable or totally unstable at the origin. Therefore, it poses a restriction to the proposed nonlinear observer design method.

Assumption 2. Denoting by H the $1 \times n$ matrix: $H = \partial h(0)/\partial x$, it is assumed that the following $n \times n$ matrix O:

$$O = \begin{bmatrix} H \\ HF \\ \cdot \\ \cdot \\ HF^{n-1} \end{bmatrix} \tag{3}$$

has rank n.

This assumption states that the linearization of (1) around the origin $x = 0$ is observable.

Motivated by Luenberger's original ideas on the linear observer design problem [12], the proposed approach will try to build a dynamic system which, driven by the output measurement y, is capable of reconstructing a nonlinear invertible function $T(x)$ of the state vector x. In particular, the following definition is proposed, which might be viewed as a generalization of Luenberger's early notion of observers to nonlinear systems [12]:

Definition 1. A dynamic system:

$$\dot{z} = \phi(z, y) \tag{4}$$

with $z \in R^n$, $y \in R$, $\phi : R^n \times R \to R^n$, is called an observer for (1), if there exists a locally invertible (around the origin) map $T(x)$ with $T : R^n \to R^n$ and $T(0) = 0$, such that if $z(0) = T(x(0))$ with $x(0)$ near zero, then $z(t) = T(x(t))$ for all $t > 0$. In particular, when T is the identity map, (4) is called the identity observer for (1).

An immediate consequence of the above definition is that, if ϕ and T are related via:

$$\frac{\partial T}{\partial x} f(x) = \phi(T(x), h(x)), \quad \phi(T(0), 0) = 0 \tag{5}$$

then the system (4) is an observer for the system (1), whenever $T(x)$ is invertible. In the special case of an identity observer, condition (5) collapses to

$$f(x) = \phi(x, h(x)), \quad \phi(0,0) = 0. \tag{6}$$

Condition (6) is the well-known mathematical requirement that the standard definition of the nonlinear observer entails:

Definition 2 (Tsinias [16]). Consider the dynamic system:

$$\dot{z} = \phi(z, y) \tag{7}$$

with $z \in R^n$ and the region:

$$M = \{(x, z) \in R^n \times R^n : x = z\}. \tag{8}$$

Assume that for any neighborhood U of M there is a neighborhood V, such that the solutions of the composite system (1) and (7) starting in V, remain in U for all times $t \geqslant 0$. Furthermore, assume that there is a neighborhood W of M such that for all solutions of the composite system (1) and (7) starting in W, their distance from M goes asymptotically to zero. Then, the dynamic system (7) is called an observer for system (1). Notice, that in such a case, M is a positively invariant set, and therefore Eq. (6) holds true [16].

Remark 1. Contrary to Definition 2, which imposes a stability requirement on the observer (7), Definition 1 does not impose such a requirement, in the same spirit as the original definition of the linear observer provided by Luenberger [12]. The stability requirement will be imposed at the observer design stage, where ϕ will be a priori selected to ensure asymptotic stability of the observer (4).

2.1. An associated system of first-order PDEs

Let us now return to the nonlinear observer of Definition 1. Notice that the vector field $\phi(z, y)$ in the observer dynamic equations (4) may be arbitrarily selected, as long as the system of PDEs (5) admits an invertible solution $T(x)$. For simplicity reasons, it would be more practical to request linear observer dynamics in the transformed states z by choosing

$$\phi(z, y) = Az + by, \tag{9}$$

where A, b are constant matrices with appropriate dimensions. In this way stability of the observer (4) can be enforced, with arbitrarily selected eigenvalues for the matrix A. Under the above selection, condition (5) will be satisfied for $T(x) = w(x)$, where $w : R^n \to R^n$ is the solution of the system of PDEs

$$\frac{\partial w}{\partial x} f(x) = Aw + bh(x), \quad w(0) = 0. \tag{10}$$

Notice that the above set of first-order PDEs has a common principal part [5], that consists of the components $f_i(x)$ $(i = 1, \ldots, n)$ of the $f(x)$ vector field. Moreover, the origin is a *characteristic point* for the above system of PDEs, since the principal part vanishes at $x = 0$. In order to solve the above system of PDEs (10), we have to distinguish two cases, depending on the region of validity of the analysis relative to the characteristic point $x = 0$.

- *Case* 1: If a solution of (10) is sought in a region of state-space that does not contain the characteristic point $x = 0$, the existeness and uniqueness conditions of the Cauchy–Kovalevskaya theorem are satisfied and therefore, the solution can be found by applying the well-known method of characteristics [5]. For example, if the inherently transient behavior of a batch chemical reactor is considered, then it will inevitably prevent the trajectories of the corresponding dynamic model to reach the $x = 0$ characteristic point, in any finite time interval [9]. Letting A_i, b_i $(i = 1, \ldots, p)$ be the rows of the A, b matrices respectively, and since the above system of PDEs has a common principal part, the characteristic system of ordinary differential equations of (10) is of the following form [5]:

  $$\frac{dx_i}{ds} = f_i(x), \qquad \frac{dw_j}{ds} = A_j w + b_j h(x) \tag{11}$$

  with $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Once arbitrary initial Cauchy-data on a non-characteristic surface have been specified, the integration of Eq. (11) provides the family of integral curves of Eq. (10), that generates the appropriate integral surface, in accordance to the general method of characteristics [5].
- *Case* 2: If a solution of Eq. (10) is sought in a neighborhood of the characteristic point $x = 0$, the

conditions of the Cauchy–Kovalevskaya theorem are not satisfied [5]. However, for the specific structure of the system (10), *Lyapunov's auxiliary theorem* can be employed to guarantee the existeness and uniqueness of the solution.

The theoretical results in the next section focus on Case 2. However, with minor modifications, they would be applicable to Case 1 as well.

We now proceed with the presentation of Lyapunov's auxiliary theorem, that forms the basis for the development of the proposed nonlinear observer design method.

Lyapunov's Auxiliary Theorem [13]. *Consider the following first-order system of quasi-linear partial differential equations*:

$$\frac{\partial w}{\partial x}\phi(x,w) = \psi(x,w), \quad w(0)=0 \tag{12}$$

with

$$\phi(0,0)=0, \qquad \psi(0,0)=0, \qquad \frac{\partial \phi}{\partial w}(0,0)=0, \tag{13}$$

where $w: R^n \to R^p$ *is the unknown vector field of* Eq. (12), *and* $\phi(x,w): R^n \times R^p \to R^n$, $\psi(x,w): R^n \times R^p \to R^p$ *are analytic vector fields. It is assumed, that the eigenvalues* k_i $(i=1,\dots,n)$ *of the* $n\times n$ *matrix* $\partial\phi(0,0)/\partial x$ *satisfy the following condition*:

$$0 \notin CH\{k_1, k_2, \dots, k_n\} \tag{14}$$

and are not related to the eigenvalues λ_i $(i=1,\dots,p)$ *of the* $p\times p$ *matrix* $\partial\psi(0,0)/\partial w$ *through any equation of the type*:

$$\sum_{i=1}^{n} m_i k_i = \lambda_j \tag{15}$$

$(j=1,\dots,p)$, *where all the* m_i *are non-negative integers that satisfy the condition*:

$$\sum_{i=1}^{n} m_i > 0. \tag{16}$$

Then, the above first-order system of PDEs (12), *with initial condition* $w(0)=0$, *admits a unique analytic solution* w *in a neighborhood of* $x=0$.

Let us now consider the linear case, where $\phi(x,w)=Fx$ and $\psi(x,w)=Aw+bHx$, with F,A,b,H being constant matrices with dimensions: $n\times n$, $n\times n$, $1\times n$ and $n\times 1$, respectively. Then, the unique solution of Eq. (12) is: $w=Wx$, where W is the solution of

$$WF - AW = bH. \tag{17}$$

As proven in [3, 12], the above matrix equation (17) admits a unique solution W, as long as the F,A matrices do not have common eigenvalues, and this is guaranteed by the assumptions of Lyapunov's auxiliary theorem.

3. Main results

We are now in a position to present the main theoretical results of the present paper on the nonlinear observer design problem.

Theorem 1. *Suppose that for the dynamic system* (1), *Assumptions* 1 *and* 2 *hold. Consider the nth order dynamic system*:

$$\dot{z} = Az + by, \tag{18}$$

where the $n\times n$ *matrix* A *is Hurwitz, its eigenvalues are not related to the eigenvalues of* F *through any equations of the type* (15), (16), *and that the pair* $\{A,b\}$ *is chosen to be controllable. Then, there exists a locally invertible analytic nonlinear map* $z=T(x)$, *that makes the dynamic system* (18) *an observer for system* (1) *in the sense of Definition* 1.

Proof. In view of Definition 1, if the nonlinear map $T(x)$ satisfies the following set of PDEs:

$$\frac{\partial T}{\partial x} f(x) = AT + bh(x) \tag{19}$$

then $z=T(x)$ follows the dynamics (18). The above condition is exactly the system of first-order PDEs (10), associated with the original system (1). Under the assumptions stated, the system of PDEs (10) admits a unique analytic solution $w=T(x)$, according to Lyapunov's auxiliary theorem. Furthermore, if one applies the linear differential operator $\partial/\partial x$ to both sides of Eq. (19) and evaluates all the resulting quantities at the equilibrium point $x=0$, one obtains:

$$\frac{\partial T}{\partial x}(0)F = A\frac{\partial T}{\partial x}(0) + bH. \tag{20}$$

The above is a matrix equation of the type (17), with unknown the Jacobian: $\partial T(0)/\partial x$ evaluated at $x=0$.

According to a well-known result, the stated assumptions: observability of the pair $\{H, F\}$ and controllability of the pair $\{A, b\}$, represent a set of necessary and sufficient conditions for the solution of Eq. (20) to be nonsingular [3, 12], and therefore the solution $T(x)$ of Eq. (19) to be locally invertible around $x = 0$. □

Remark 2. The above result may be extended for the case of multiple-output systems as well [10].

Remark 3. The above result generalizes to the case where the requested z-dynamics is of the form:

$$\dot{z} = Az + G(y), \tag{21}$$

where $G(y)$ is locally analytic around the origin and $G(0) = 0$. The result remains unaffected, as long as $\partial G(0)/\partial y \neq 0$. In this case, the matrix $\partial G(0)/\partial y$ replaces b in the proof. Also, all the results that will follow remain valid under this generalization.

Remark 4. It should be emphasized that the above linearization approach is fundamentally different from the one adopted by Krener and Isidori [11]. As a first step, the authors in [11] consider a nonlinear coordinate transformation, to transform the original system (1) into a linear one (with the addition of an output injection term), which has a linear output map as well:

$$\dot{z} = Az + \psi(y), \quad y = cz. \tag{22}$$

The design of the observer is then completed in a second step, where a standard linear Luenberger observer is built for the transformed system (22) [11]:

$$\dot{\hat{z}} = A\hat{z} - L(y - c\hat{z}) + \psi(y), \tag{23}$$

where L is the constant observer gain and $\hat{z}$ the estimate of the transformed vector z. Finally, the estimate $\hat{x}$ of the original state vector x is recovered, through the inverse transformation employed in the first step of the design procedure. In the above *two-step approach*, the requirement of a linear output map in the transformed system (22), represents the main mathematical reason for the appearance of a very restrictive set of conditions (involutivity conditions) in their proposed solution [11]. In the present work, the state observer is viewed as a dynamic system that is driven by the measured output variable, and that can be designed through an appropriate coordinate transformation directly and *in only one step*, without the redundant requirement of linearity of the output map of the transformed linear system (18). Indeed, after solving the linear system of first-order PDEs (10), the transformed output map becomes: $y = h(T^{-1}(z))$ and is, in general, nonlinear.

Theorem 2. *Let $w = T(x)$ be an invertible solution of* Eq. (10). *The dynamic system*

$$\dot{\hat{x}} = f(\hat{x}) + \left[\frac{\partial T}{\partial \hat{x}}(\hat{x})\right]^{-1} b(y - h(\hat{x})) \tag{24}$$

is an identity observer for the original system (1), *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(\hat{x}) - T(x)) = A(T(\hat{x}) - T(x)). \tag{25}$$

Proof. One obtains:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(T(\hat{x}) - T(x)) &= \frac{\partial T}{\partial \hat{x}}\dot{\hat{x}} - \frac{\partial T}{\partial x}\dot{x} \\ &= \frac{\partial T}{\partial \hat{x}} f(\hat{x}) - \frac{\partial T}{\partial x} f(x) + \frac{\partial T}{\partial \hat{x}}\left[\frac{\partial T}{\partial \hat{x}}\right]^{-1} b(y - h(\hat{x})) \\ &= AT(\hat{x}) - AT(x) + bh(\hat{x}) + b(y - h(\hat{x})) - bh(x) \\ &= A(T(\hat{x}) - T(x)).\end{aligned} \tag{26}$$

In summary, the proposed nonlinear observer is of the form

$$\dot{\hat{x}} = f(\hat{x}) + L(\hat{x})(y - h(\hat{x})) \tag{27}$$

with a nonlinear gain

$$L(x) = \left[\frac{\partial T}{\partial x}(x)\right]^{-1} b, \tag{28}$$

where $w = T(x)$ is the solution of Eq. (10). Because A is Hurwitz and $T(x)$ is a continuous invertible map, condition (25) guarantees that $\hat{x}$ asymptotically approaches x. Notice the state-dependent nonlinear gain $L(x)$ of the proposed observer (24), as opposed to the constant gain of the linear case. Indeed, in the linear case, where: $f(x) = Fx$ and $h(x) = Hx$, the problem of solving the system of PDEs (10) reduces to the problem of solving the matrix equation:

$$TF - AT = bH. \tag{29}$$

Under the stated assumptions, the corresponding system of PDEs (10) admits a unique solution: $w = Tx$, with T being the unique invertible solution of Eq. (29). Therefore, the constant gain $L = T^{-1}b$ is obtained, and the proposed observer simply becomes the well-known linear Luenberger observer [12]. □

Example. Consider the following nonlinear system:

$$\dot{x}_1 = -x_2,$$
$$\dot{x}_2 = (x_1+1)(x_2+1)-1,$$
$$y = -x_2^2 + x_1(x_1+2)(x_2+1). \tag{30}$$

The above system is locally observable at the equilibrium point $(0,0)$ and the eigenvalues of the Jacobian matrix:

$$F = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

are $k_1 = \frac{1}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}$, $k_2 = \frac{1}{2} + \frac{1}{2}\sqrt{3}\mathrm{i}$. We select the Hurwitz matrix:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\mathrm{i}$, $\lambda_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}$, such that conditions of the type (15), (16) are avoided. Moreover, we choose $b = (\,0 \quad 1\,)^{\mathrm{T}}$ so that the pair $\{A, b\}$ is controllable. Since all the assumptions are now valid, the first-order system of PDEs

$$\frac{\partial T_1}{\partial x_1}(-x_2) + \frac{\partial T_1}{\partial x_2}((x_1+1)(x_2+1)-1) = T_2,$$

$$\frac{\partial T_2}{\partial x_1}(-x_2) + \frac{\partial T_2}{\partial x_2}((x_1+1)(x_2+1)-1)$$
$$= -T_1 - T_2 - x_2^2 + x_1(x_1+2)(x_2+1) \tag{31}$$

admits an invertible analytic solution, in a neighborhood of the equilibrium point $(0,0)$. Indeed, assuming a certain polynomial form for T_1 and T_2, we inserted their expressions into (31) and evaluated the corresponding coefficients by equating terms of the same order. We found:

$$T_1 = x_1 + x_2, \qquad T_2 = x_1 x_2 + x_1. \tag{32}$$

Considering the Jacobian:

$$\frac{\partial T}{\partial x} = \begin{pmatrix} 1 & 1 \\ 1+x_2 & x_1 \end{pmatrix} \tag{33}$$

we may conclude that it is indeed invertible at $(0,0)$. The nonlinear gain $L(x)$ of the proposed observer is

$$L(x_1,x_2) = \left(\frac{\partial T}{\partial x}\right)^{-1} b = \frac{1}{x_1 - x_2 - 1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{34}$$

Hence, the dynamic equations for the proposed nonlinear identity observer become

$$\dot{\hat{x}}_1 = -\hat{x}_2 - \frac{1}{\hat{x}_1 - \hat{x}_2 - 1}\cdot (y + \hat{x}_2^2 - \hat{x}_1(\hat{x}_1+2)(\hat{x}_2+1)),$$

$$\dot{\hat{x}}_2 = (\hat{x}_1+1)(\hat{x}_2+1) - 1 + \frac{1}{\hat{x}_1 - \hat{x}_2 - 1}\cdot (y + \hat{x}_2^2 - \hat{x}_1(\hat{x}_1+2)(\hat{x}_2+1)). \tag{35}$$

4. Series solution of the system of PDEs

In order to be able to make practical use of the proposed nonlinear observer design methodology, one must provide a solution scheme for the associated system of first-order linear PDEs (10). Note that the method of characteristics is not applicable because the aforementioned system of PDEs (10) is singular. However, since $f(x)$, $h(x)$ and the solution $T(x)$ are locally analytic around the reference equilibrium point, it is possible to calculate the solution $T(x)$ in the form of a multivariate Taylor series around the reference equilibrium point. The method involves expanding $f(x)$, $h(x)$ and the unknown $T(x)$ in a Taylor series and equating the Taylor coefficients of both sides of the PDEs. This procedure leads to recursion formulas, through which one can calculate the Nth order Taylor coefficient of $T(x)$, given the Taylor coefficients of $T(x)$ up to the order $N-1$.

In the derivation of the recursion formulas, it is convenient to use the following tensorial notation:

(a) The entries of a constant matrix A are represented as a_i^j, where the subscript i refers to the corresponding row and the superscript j to the corresponding column of the matrix.

(b) The partial derivatives of the μ component $f_\mu(x)$ of a vector field $f(x)$ at $x=0$ are denoted as follows:

$$f_\mu^i = \frac{\partial f_\mu}{\partial x_i}(0), \qquad f_\mu^{ij} = \frac{\partial^2 f_\mu}{\partial x_i \partial x_j}(0),$$
$$f_\mu^{ijk} = \frac{\partial^3 f_\mu}{\partial x_i \partial x_j \partial x_k}(0), \tag{36}$$

etc.

(c) The standard summation convention, where repeated upper and lower tensorial indices are summed up.

With this notation the lth component $T_l(x)$ of the unknown solution $T(x)$ is expanded in a multivariate Taylor series as follows:

$$T_l(x) = \frac{1}{1!}T_l^{i_1}x_{i_1} + \frac{1}{2!}T_l^{i_1 i_2}x_{i_1}x_{i_2} + \cdots + \frac{1}{N!}T_l^{i_1 i_2 \cdots i_N}x_{i_1}x_{i_2}\cdots x_{i_N} + \cdots. \quad (37)$$

Substituting the Taylor series expansions of $T(x)$, $f(x)$ and $h(x)$ into Eq. (10) and matching the Taylor coefficients, the following relation for the Nth order terms may be obtained:

$$\sum_{L=0}^{N-1}\sum_{\binom{N}{L}} T_l^{\mu i_1\cdots i_L} f_\mu^{i_{L+1}\cdots i_N} = a_l^\mu T_\mu^{i_1\cdots i_N} + b_l h^{i_1\cdots i_N}, \quad (38)$$

where $i_1,\ldots,i_N = 1,\ldots,n$ and $l = 1,\ldots,n$. Note that the second summation symbol in Eq. (38) should be regarded as summing up the relevant quantities over the $\binom{N}{L}$ possible combinations of the indices $(i_1,\ldots,i_N)$. Eqs. (38) represent a set of linear algebraic equations in the unknown coefficients $T_\mu^{i_1,\ldots,i_N}$.

The series solution of the PDEs (10) around an equilibrium point of interest, may be accomplished in an automatic fashion, by exploiting the computational capabilities and commands of MAPLE. Specifically, an efficient MAPLE code to automatically generate the various coefficients of the multivariate Taylor series expansion of the unknown solution of Eq. (10) has been developed [10].

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