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On the relationship between parity space and H_2 approaches to fault detection

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Abstract

Parity space approach and H_2 approach are two important fault detection approaches. This paper studies the relationship between these two approaches, which reveals frequency domain characteristics of the optimal solution of the parity space approach on the one side and provides a numerical solution of the H_2-optimal design of residual generators on the other side.

Keywords: Fault detection; Discrete-time systems; Parity space approach; Frequency domain approach; Robustness

1. Introduction

Parity space approach and H_2 approach are two commonly used approaches for designing robust fault detection systems [1,9,14]. The former is initially proposed by [3,4] and has been extensively studied since then [2,5–7,10,12,13,15]. The latter is proposed by [8].

In this paper, some insight will be shed on the relationship between these two approaches, which simultaneously enhances our understanding about the optimal solution of the parity space approach and provides us a numerical way to calculate the H_2-optimal solution of residual generator design. It is proven that the optimal parity vector approximates the H_2-optimal residual generator and thus it is a bandpass filter whose bandwidth will become narrower as the order of the parity relation increases.

The paper is organized as follows. First, the parity space approach is briefly reviewed in Section 2. Then, Section 3 gives the optimal solution of the H_2 approach in the context of discrete-time systems. The relationship between the parity space approach and the H_2 approach is studied in Section 4. Finally, the results are illustrated by an example in Section 5.

2. Brief review of the parity space approach

In this contribution, we consider linear discrete time-invariant systems described by

$$x(k+1) = Ax(k) + Bu(k) + E_d d(k) + E_f f(k), \tag{1}$$

$$y(k) = Cx(k) + Du(k) + F_d d(k) + F_f f(k), \tag{2}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{k_u}$, $y \in \mathbf{R}^m$, $d \in \mathbf{R}^{k_d}$, $f \in \mathbf{R}^{k_f}$ denote the vector of states, control inputs, measurement outputs, unknown disturbances and faults to be detected, respectively. A, B, C, D, E_d, E_f, F_d and F_f are known matrices of appropriate dimensions. It is assumed that (C, A) is observable.

A parity relation based residual generator can be constructed as [1,10,12,13,15]

$$r_s(k) = v_s(y_s(k) - H_{u,s}u_s(k)), \tag{3}$$

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where $r_s \in \mathbf{R}$ is the residual signal, row vector $v_s \in \mathbf{R}^{m(s+1)}$ is the parity vector which satisfies

$$v_s \in P_s, \quad P_s = \{v_s \mid v_s H_{o,s} = 0\}. \tag{4}$$

P_s is called parity space, s denotes the order of the parity relation, and

$$y_s(k) = \begin{bmatrix} y(k-s) \\ y(k-s+1) \\ \vdots \\ y(k) \end{bmatrix}, \quad u_s(k) = \begin{bmatrix} u(k-s) \\ u(k-s+1) \\ \vdots \\ u(k) \end{bmatrix},$$

$$H_{u,s} = \begin{bmatrix} D & O & \cdots & O \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}B & \cdots & CB & D \end{bmatrix}, \quad H_{o,s} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix}. \tag{5}$$

The dynamics of residual generator (3) is governed by

$$r_s(k) = v_s(H_{d,s}d_s(k) + H_{f,s}f_s(k)), \tag{6}$$

where

$$d_s(k) = \begin{bmatrix} d(k-s) \\ d(k-s+1) \\ \vdots \\ d(k) \end{bmatrix}, \quad f_s(k) = \begin{bmatrix} f(k-s) \\ f(k-s+1) \\ \vdots \\ f(k) \end{bmatrix},$$

$$H_{d,s} = \begin{bmatrix} F_d & O & \cdots & O \\ CE_d & F_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_d & \cdots & CE_d & F_d \end{bmatrix},$$

$$H_{f,s} = \begin{bmatrix} F_f & O & \cdots & O \\ CE_f & F_f & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_f & \cdots & CE_f & F_f \end{bmatrix}. \tag{7}$$

If there exists such a parity vector $v_s \in P_s$, which makes

$$v_s H_{d,s} = 0, \quad v_s H_{f,s} \neq 0 \tag{8}$$

then the residual is said to be fully decoupled from the unknown disturbances. However, such a full decoupling is seldom achievable in practice, since there are usually less measurements than unknown disturbances in the system. And in these cases, a suitable trade-off between the sensitivity of the residual generator to the faults and its robustness to the disturbances is necessary. To the aim of evaluating the performance of the residual generator, J_s defined by

$$J_s = \frac{v_s H_{d,s} H_{d,s}^{\mathrm{T}} v_s^{\mathrm{T}}}{v_s H_{f,s} H_{f,s}^{\mathrm{T}} v_s^{\mathrm{T}}} \tag{9}$$

is one of the most often used indexes [5,7,15]. So the optimal design of the residual generator consists in finding the optimal parity vector $v_s \in P_s$ which solves the optimization problem

$$\min_{v_s \in P_s} J_s = \min_{v_s \in P_s} \frac{v_s H_{d,s} H_{d,s}^{\mathrm{T}} v_s^{\mathrm{T}}}{v_s H_{f,s} H_{f,s}^{\mathrm{T}} v_s^{\mathrm{T}}}. \tag{10}$$

Ding et al. [5] have proven that with the increase of the order s of the parity relation, the performance of the residual generator in the sense of (9) will also be improved, i.e.

$$\min_{v_{s+1} \in P_{s+1}} J_{s+1} < \min_{v_s \in P_s} J_s. \tag{11}$$

Because $\min_{v_s \in P_s} J_s$ is lower bounded by 0 and decreases with respect to s, the limit $\lim_{s\to\infty} \min_{v_s \in P_s} J_s$ exists and

$$\lim_{s\to\infty} \min_{v_s \in P_s} J_s = \min_{s} \min_{v_s \in P_s} J_s. \tag{12}$$

3. Optimal solution of the H_2 approach

The H_2 approach is originally proposed in [8] in the context of linear continuous-time systems. In this section, a discrete-time version of this approach will be presented.

Given system (1)–(2), use

$$G_u(z) = C(zI - A)^{-1}B + D,$$
$$G_d(z) = C(zI - A)^{-1}E_d + F_d,$$
$$G_f(z) = C(zI - A)^{-1}E_f + F_f$$

to denote the transfer function matrices from u, d and f to y, respectively. It is well-known that all linear time-invariant residual generators can be expressed by [9]

$$r(z) = R(z)(\hat{M}_u(z)y(z) - \hat{N}_u(z)u(z)), \tag{13}$$

where $R(z) \in \mathbf{RH}_\infty$ is called post-filter and arbitrarily selectable, $(\hat{M}_u(z), \hat{N}_u(z))$ is a left coprime factorization of $G_u(z)$, i.e. $G_u(z) = \hat{M}_u^{-1}(z)\hat{N}_u(z)$. $\hat{M}_u(z)$ and $\hat{N}_u(z)$ can be calculated as follows:

$$\hat{M}_u(z) = I - C(zI - A + LC)^{-1}L,$$
$$\hat{N}_u(z) = D + C(zI - A + LC)^{-1}(B - LD), \tag{14}$$

where L is a matrix of compatible dimensions that stabilizes $A - LC$.

The dynamics of residual generator (13) is governed by

$$r(z) = R(z)\hat{M}_u(z)(G_d(z)d(z) + G_f(z)f(z)). \tag{15}$$

In case that a full decoupling is not achievable, the H_2 performance index for the optimal design of a robust residual generator is defined as

$$\min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} \frac{\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R^*(e^{j\omega})\,\mathrm{d}\omega}{\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R^*(e^{j\omega})\,\mathrm{d}\omega}, \tag{16}$$

where the superscript $*$ denotes the conjugate transpose of the matrix and the post-filter $R(z)$ is assumed to be a vector of transfer functions.

Theorem 1. *Given system* (1)–(2), *the optimal solution to optimization problem* (16) *is*

$$R_{\mathrm{opt}}(z) = f_{\omega_0}(z)p(z), \quad J_{\mathrm{opt}} = \inf_{\omega} \sigma_{\min}(\omega) \tag{17}$$

where $f_{\omega_0}(z)$ is an ideal frequency-selective filter with the selective frequency at ω_0, which satisfies

$$f_{\omega_0}(e^{j\omega})q(e^{j\omega}) = 0, \quad \omega \neq \omega_0, \tag{18}$$

$$\int_0^{2\pi} f_{\omega_0}(e^{j\omega})q(e^{j\omega})q^*(e^{j\omega})f_{\omega_0}^*(e^{j\omega})\,\mathrm{d}\omega = q(e^{j\omega_0})q^*(e^{j\omega_0}), \quad \forall q^{\mathrm{T}}(z) \in \mathbf{RH}_2,$$

$\sigma_{\min}(\omega)$ and $p(e^{j\omega})$ are, respectively, the minimal generalized eigenvalue and corresponding eigenvector of the following generalized eigenvalue–eigenvector problem

$$p(e^{j\omega})(\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega}) - \sigma_{\min}(\omega)\hat{M}_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})) = 0 \tag{19}$$

and ω_0 is the frequency at which $\sigma_{\min}(\omega)$ achieves its minimum, i.e.

$$\sigma_{\min}(\omega_0) = \inf_{\omega} \sigma_{\min}(\omega).$$

Proof. Substituting $R_{\mathrm{opt}}(z) = f_{\omega_0}(z)p(z)$ into the performance index J and taking (18) into consideration, there is

$$J = \frac{p(e^{j\omega_0})\hat{M}_u(e^{j\omega_0})G_d(e^{j\omega_0})G_d^*(e^{j\omega_0})\hat{M}_u^*(e^{j\omega_0})p^*(e^{j\omega_0})}{p(e^{j\omega_0})\hat{M}_u(e^{j\omega_0})G_f(e^{j\omega_0})G_f^*(e^{j\omega_0})\hat{M}_u^*(e^{j\omega_0})p^*(e^{j\omega_0})}.$$

From (19), it is clear that

$$J = \sigma_{\min}(\omega_0).$$

Note that for any post-filter $R(z) \in \mathbf{RH}_\infty^{1\times m}$ and for all ω

$$R(e^{j\omega})\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R^*(e^{j\omega}) - \sigma_{\min}(\omega_0)R(e^{j\omega})\hat{M}_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega}) \times \hat{M}_u^*(e^{j\omega})R^*(e^{j\omega}) \geqslant 0$$

holds, which leads to

$$\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega}) \times R^*(e^{j\omega})\,\mathrm{d}\omega - \sigma_{\min}(\omega_0)\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega}) \times G_f(e^{j\omega})G_f^*(e^{j\omega}) \times \hat{M}_u^*(e^{j\omega})R^*(e^{j\omega})\,\mathrm{d}\omega \geqslant 0.$$

As a result, we have for any post-filter $R(z) \in \mathbf{RH}_\infty^{1\times m}$

$$J \geqslant \sigma_{\min}(\omega_0).$$

This demonstrates that $\sigma_{\min}(\omega_0)$ is indeed the optimal value and correspondingly $R_{\mathrm{opt}}(z) = f_{\omega_0}(z)p(z)$ is the optimal solution. □

Remark 1. At any frequency ω, the matrices $\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})$ and $\hat{M}_u(e^{j\omega})\,G_f(e^{j\omega})G_f^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})$ are positive semi-definite Hermitian matrices. Therefore, the generalized eigenvalues $\sigma(\omega)$ in (19) are always real [11].

Remark 2. The optimal solution to optimization problem (16) is independent of matrix L in the sense that, as long as L is stabilizing, $R_{\mathrm{opt}}(z)\hat{M}_u(z)$ and J_{opt} do not change with L.

4. Relationship between two approaches

In this section, we present the main result of this paper, the discussion on the relationship between the optimal solutions of the parity space approach and the H_2 approach.

Suppose that $\{g_d(0), g_d(1), \ldots\}$ is the impulse response of system(1)–(2) to the unknown disturbances. Apparently,

$$g_d(0) = F_d,\ g_d(1) = CE_d, \ldots, \quad g_d(s) = CA^{s-1}E_d, \ldots \tag{20}$$

The matrix $H_{d,s}$ can then be expressed in terms of the impulse response as follows

$$H_{d,s} = \begin{bmatrix} g_d(0) & O & \cdots & O \\ g_d(1) & g_d(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ g_d(s) & \cdots & g_d(1) & g_d(0) \end{bmatrix}.$$

Partition the parity vector v_s as

$$v_s = [\, v_{s,0} \quad v_{s,1} \quad \cdots \quad v_{s,s} \,],$$

where the row vector $v_{s,i} \in \mathbf{R}^m$, $i = 0, 1, \ldots s$.

Then, we have

$$v_s H_{d,s} = [\,\varphi(s) \quad \varphi(s-1) \quad \cdots \quad \varphi(0)\,],$$

where

$$\varphi(i) = \sum_{l=0}^{i} \rho_{i-l} g_d(l), \quad \rho_i = v_{s,s-i}, \quad i = 0, 1, \ldots s.$$

Let s go to infinity. It leads to

$$\lim_{s\to\infty} v_s H_{d,s} = [\,\varphi(\infty) \quad \cdots \quad \varphi(0)\,] \tag{21}$$

and in this case

$$\begin{aligned} \varphi(i) &= \sum_{l=0}^{i} \rho_{i-l} g_d(l) = \rho(i) \otimes g_d(i) \\ &= \mathscr{Z}^{-1}(P(z)G_d(z)), \end{aligned} \tag{22}$$

$$P(z) = \mathscr{Z}[\rho(i)], \quad \rho(i) = \{\rho_0, \rho_1, \ldots\}, \tag{23}$$

where $\otimes$ denotes the convolution. Eq. (23) means that $P(z)$ is the z-transform of the sequence $\{\rho_0, \rho_1, \ldots\}$.

According to the Parseval Theorem, we have

$$\begin{aligned} \lim_{s\to\infty} v_s H_{d,s} H_{d,s}^{\mathrm{T}} v_s^{\mathrm{T}} &= \sum_{i=0}^{\infty} \varphi(i)\varphi^{\mathrm{T}}(i) \\ &= \frac{1}{2\pi} \int_0^{2\pi} P(e^{j\omega}) G_d(e^{j\omega}) \\ &\quad \times G_d^*(e^{j\omega}) P^*(e^{j\omega})\, \mathrm{d}\omega. \end{aligned} \tag{24}$$

Similarly, it can be proven that

$$\begin{aligned} \lim_{s\to\infty} v_s H_{f,s} H_{f,s}^{\mathrm{T}} v_s^{\mathrm{T}} &= \frac{1}{2\pi} \int_0^{2\pi} P(e^{j\omega}) G_f(e^{j\omega}) \\ &\quad \times G_f^*(e^{j\omega}) P^*(e^{j\omega})\, \mathrm{d}\omega. \end{aligned} \tag{25}$$

On the other side, if given a residual generator (13), we can always construct a parity vector, as stated in Lemma 1.

Lemma 1. *Given system* (1)–(2) *and a residual generator* (13) *with* $R(z) \in \mathbf{RH}_\infty^{1\times m}$. *Then the row vector defined by*

$$v = [\cdots \quad \bar{C}\bar{A}\bar{B} \quad \bar{C}\bar{B} \quad \bar{D}\,], \tag{26}$$

where $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ *is the state space realization of* $R(z)\hat{M}_u(z)$, *belongs to the parity space* P_s $(s \to \infty)$.

Proof. Assume that (A_r, B_r, C_r, D_r) is a state space realization of $R(z)$. Recalling (14), we know that

$$\bar{A} = \begin{bmatrix} A - LC & O \\ -B_r C & A_r \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} L \\ B_r \end{bmatrix},$$

$$\bar{C} = [\,-D_r C \quad C_r\,], \quad \bar{D} = D_r.$$

It can be easily obtained that

$$\begin{aligned} \lim_{s\to\infty} vH_{o,s} &= \lim_{s\to\infty} [\cdots \quad \bar{C}\bar{A}\bar{B} \quad \bar{C}\bar{B} \quad \bar{D}\,] \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}, \\ &= \lim_{s\to\infty} [\cdots \quad C_r A_r B_r \quad C_r B_r \quad D_r\,] \\ &\quad \times \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix}. \end{aligned} \tag{27}$$

For a linear discrete-time system

$$\begin{aligned} \lambda(k+1) &= (A - LC)\lambda(k) \\ \delta(k) &= C\lambda(k) \end{aligned} \tag{28}$$

with any initial state vector $\lambda(0) = \lambda_0 \in \mathbf{R}^n$, apparently,

$$\begin{aligned} &\delta(0) = C\lambda_0, \\ &\delta(1) = C(A-LC)\lambda_0, \\ &\delta(2) = C(A-LC)^2\lambda_0, \ldots . \end{aligned}$$

Since $R(z) \in \mathbf{RH}_\infty^{1\times m}$ and L is selected to ensure the stability of $A - LC$, the cascade connection of system (28) and $R(z)$ is stable. So

$$\lim_{k\to\infty} \mathscr{Z}^{-1}\{R(z)\delta(z)\} = 0.$$

Note that

$$\begin{aligned} &\lim_{k\to\infty} \mathscr{Z}^{-1}\{R(z)\delta(z)\} \\ &\quad = \lim_{s\to\infty} [\cdots \quad C_r A_r B_r \quad C_r B_r \quad D_r\,] \\ &\qquad \times \begin{bmatrix} C\lambda_0 \\ C(A-LC)\lambda_0 \\ C(A-LC)^2\lambda_0 \\ \vdots \end{bmatrix}, \end{aligned}$$

we get

$$\lim_{s\to\infty} [\cdots \quad C_r A_r B_r \quad C_r B_r \quad D_r\,] \times \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix} \lambda_0 = 0,$$

for any initial state vector $\lambda_0 \in \mathbf{R}^n$. Thus it can be concluded that

$$\lim_{s\to\infty} [\cdots \quad C_r A_r B_r \quad C_r B_r \quad D_r\,] \times \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix} = 0.$$

At last, from (27) we obtain

$$\lim_{s\to\infty} vH_{o,s} = 0,$$

i.e. the vector v defined by (26) belongs to the parity space P_s ($s \to \infty$). Lemma 1 is thus proven. □

It is of interest to note that the vector v is indeed composed of the impulse response of the residual generator $R(z)\hat{M}_u(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B}$, which is given by $\{\bar{D}, \bar{C}\bar{B}, \bar{C}\bar{A}\bar{B}, \bar{C}\bar{A}^2\bar{B}, \ldots\}$.

Based on the above analysis, the following theorem can be obtained.

Theorem 2. *Given system* (1)–(2) *and assume that* $v_{s,\mathrm{opt}}$, $J_{s,\mathrm{opt}}$ *and* $R_{\mathrm{opt}}(z)$, J_{opt} *are the optimal solutions of optimization problems*

$$J_{s,\mathrm{opt}} = \min_{v_s \in P_s} J_s = \min_{v_s \in P_s} \frac{v_s H_{d,s} H_{d,s}^{\mathrm{T}} v_s^{\mathrm{T}}}{v_s H_{f,s} H_{f,s}^{\mathrm{T}} v_s^{\mathrm{T}}} = \frac{v_{s,\mathrm{opt}} H_{d,s} H_{d,s}^{\mathrm{T}} v_{s,\mathrm{opt}}^{\mathrm{T}}}{v_{s,\mathrm{opt}} H_{f,s} H_{f,s}^{\mathrm{T}} v_{s,\mathrm{opt}}^{\mathrm{T}}}, \tag{29}$$

$$J_{\mathrm{opt}} = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} \frac{\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R^*(e^{j\omega})\,\mathrm{d}\omega}{\int_0^{2\pi} R(e^{j\omega})\hat{M}_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R^*(e^{j\omega})\,\mathrm{d}\omega},$$
$$= \frac{\int_0^{2\pi} R_{\mathrm{opt}}(e^{j\omega})\hat{M}_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R_{\mathrm{opt}}^*(e^{j\omega})\,\mathrm{d}\omega}{\int_0^{2\pi} R_{\mathrm{opt}}(e^{j\omega})\hat{M}_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})\hat{M}_u^*(e^{j\omega})R_{\mathrm{opt}}^*(e^{j\omega})\,\mathrm{d}\omega}, \tag{30}$$

respectively. Then

$$\lim_{s\to\infty} J_{s,\mathrm{opt}} = J_{\mathrm{opt}}, \tag{31}$$

$$P(z) = R_{\mathrm{opt}}(z)\hat{M}_u(z), \tag{32}$$

where

$$P(z) = \mathscr{Z}[\rho(i)],$$
$$\rho(i) = \{v_{s\to\infty,\mathrm{opt},s}, v_{s\to\infty,\mathrm{opt},s-1}, \ldots, v_{s\to\infty,\mathrm{opt},0}\}. \tag{33}$$

Proof. Let $v_{s\to\infty,\mathrm{opt}}$ denote the optimal solution of optimization problem (29) as $s \to \infty$, then it follows from (12), (23)–(25) that for any left coprime factorization of $G_u(z) = \hat{M}_u^{-1}(z)\hat{N}_u(z)$, the post-filter $R_o(z)$ given by

$$R_o(z) = P(z)\hat{M}_u^{-1}(z),$$

where $P(z)$ is defined by (33), leads to

$$J|_{R(z)=R_o(z)} = \lim_{s\to\infty} J_{s,\mathrm{opt}} = \lim_{s\to\infty} \min_{v_s\in P_s} J_s = \min_s \min_{v_s\in P_s} J_s \geqslant \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J. \tag{34}$$

We now demonstrate that

$$J|_{R(z)=R_o(z)} = J_{\mathrm{opt}} = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J. \tag{35}$$

Suppose that (35) does not hold. Then, the optimal solution of optimization problem (30), denoted by $R_c(z) \in \mathbf{RH}_\infty^{1\times m}$ and different from $R_o(z)$, should lead to

$$J|_{R(z)=R_c(z)} = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J < J|_{R(z)=R_o(z)}. \tag{36}$$

According to Lemma 1, we can find a parity vector $v \in P_s$ whose components are just a re-arrangement of the impulse response of $R_c(z)\hat{M}_u(z)$. Moreover, because of (23)–(25), we have

$$J_s|_{v_s=v} = J|_{R(z)=R_c(z)}. \tag{37}$$

As a result, it follows from (34), (36) and (37) that

$$J_s|_{v_s=v} < \min_s \min_{v_s\in P_s} J_s$$

which is an obvious contradiction. Thus we can conclude that

$$J_{\mathrm{opt}} = \min_{R(z)\in \mathbf{RH}_\infty^{1\times m}} J = J|_{R(z)=R_o(z)} = \lim_{s\to\infty} J_{s,\mathrm{opt}}$$

and

$$R_o(z) = P(z)\hat{M}_u^{-1}(z) := R_{\mathrm{opt}}(z)$$

solve optimization problem (30). Theorem 2 is thus proven. □

Theorem 2 gives a deeper insight into the relationship between the parity space approach and the H_2 approach and reveals some very interesting facts when the order of the parity relation s increases:

- The optimal performance index $J_{s,\mathrm{opt}}$ of the parity space approach converges to a limit which is just the optimal performance index J_{opt} of the H_2 approach.
- There is a one-to-one relationship between the optimal solutions of optimization problems (29) and (30) when the order of the parity relation $s \to \infty$. Since $R_{\mathrm{opt}}(z)$ is a band-limited filter, the frequency response of $v_{s\to\infty,\mathrm{opt}}$ is also band-limited.

The above result can be applied in several ways, for instance:

- For multi-dimensional systems, the optimal solution of the H_2 approach can be approximately computed by at first calculating the optimal solution of the parity space approach with a high order s and then doing the z-transform of the optimal parity vector. It is worth noticing that numerical problem may be met for some systems, especially when A is unstable.

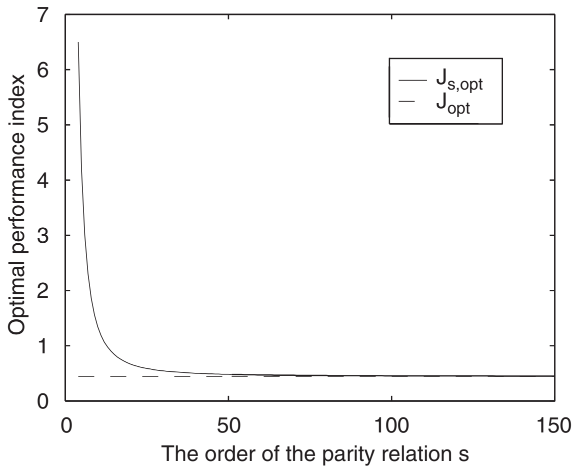

Fig. 1. The change of the optimal performance index $J_{s,\mathrm{opt}}$ with respect to s.

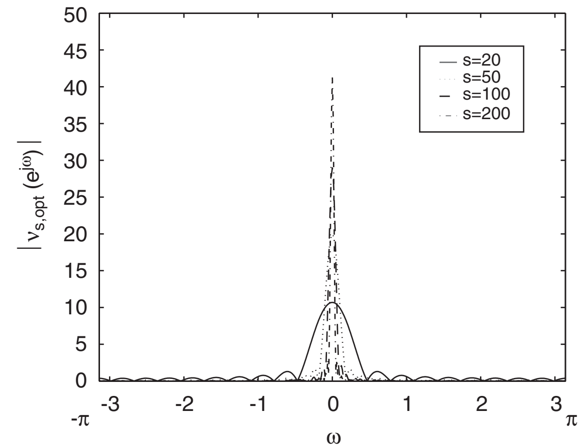

Fig. 2. The frequency response of the optimal parity vector $v_{s,\mathrm{opt}}$ with respect to s.

- In the parity space approach, a high order s will improve the performance index $J_{s,\mathrm{opt}}$ but, on the other side, increase the online computational effort. To determine a suitable trade-off between performance and implementation effort, the optimal performance index J_{opt} of the H_2 approach can be used as a reference value.
- Based on the property that the frequency response of $v_{s\to\infty,\mathrm{opt}}$ is band-limited, advanced parity space approaches can be developed to achieve both a good performance and a low order parity vector. For instance, in Refs. [17,16] infinite impulse response (IIR) filter and wavelet transform have been introduced, respectively, to design optimized parity vector of low order and good performance.

5. Numerical example

Given a discrete-time system modelled by (1)–(2), where

$$A=\begin{bmatrix}1 & -1.30\\ 0.25 & -0.25\end{bmatrix},\quad B=\begin{bmatrix}2\\ 1\end{bmatrix},\quad C=[0\quad 1],$$

$$E_d=\begin{bmatrix}0.4\\ 0.5\end{bmatrix},\quad E_f=\begin{bmatrix}0.6\\ 0.1\end{bmatrix},\quad D=F_d=F_f=0. \qquad (38)$$

As system (38) is stable, matrix L in (14) can be selected to be zero matrix and thus $\hat{M}_u(z)$ is an identity matrix. To solve the generalized eigenvalue–eigenvector problem (19) to get ω_0 that achieves $\sigma_{\min}(\omega_0)=\inf_\omega \sigma_{\min}(\omega)$, note that

$$\sigma_{\min}(\omega)=\frac{0.41-0.4\cos\omega}{0.0125+0.01\cos\omega}.$$

Therefore, the optimal performance index of the H_2 approach is $J_{\mathrm{opt}}=0.4444$ and the selective frequency is $\omega_0=0$.

Fig. 1 demonstrates the change of the optimal performance index $J_{s,\mathrm{opt}}$ with respect to the order of the parity relation s. From the figure it can be seen that $J_{s,\mathrm{opt}}$ decreases with the increase of s and, moreover, $J_{s,\mathrm{opt}}$ converges to J_{opt} when $s\to\infty$. Fig. 2 shows the frequency responses of the optimal parity vector $v_{s,\mathrm{opt}}$ when s is chosen as 20, 50, 100 and 200, respectively. We see that the bandwidth of the frequency response of $v_{s,\mathrm{opt}}$ becomes narrower and narrower with the increase of s.

6. Conclusion

The relationship between the parity space approach and the H_2 approach to fault detection of linear discrete time-invariant systems has been discussed in this paper. It is shown that with the increase of the order of the parity relation s, the optimal performance index of the parity space approach converges to that of the H_2 approach, and the frequency response of the optimal parity vector also converges to the optimal post-filter $R_{\mathrm{opt}}(z)$ in the H_2 approach. This result not only leads to a numerical solution of the H_2 optimal design but also provides theoretical basis for developing advanced parity space approaches.

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