configuration. It is evident that as γ_{2_0} decreases (i.e., the arm straightens), the coupling between the joints becomes more severe, eventually resulting in an unstable system.

V. Conclusions

We have established the feasibility, at least in principle, of decoupling the dynamics of a rigid-body model of a manipulator arm system. The advantages of the open-loop synthesis in comparison with the classical feedback solution [7] are two-fold: there is no need for complete state feedback; in addition to being conceptually simple, the open-loop synthesis is directly compatible with the IJC design philosophy adopted in most manipulator control systems.

Further work in this area includes a sensitivity study on the effect of parameter variation on decoupling. It is known that a large class of feedback decoupling solutions are structurally unstable [8]: even small parameter variations may result in a breakdown of decoupling. Hence, more future effort should be devoted to the study of approximate and adaptive decoupling [9]. The open-loop design studied here presents a feasible approach to adaptive decoupling in that most of the parameters in the synthesis are known analytic functions of the arm or motor/gearbox parameters. Problems in these and other areas, such as the decoupling of manipulators with flexible arms, will continue to demand our attention in the future.

Appendix
An IJC Design

From (16) and (22), the closed-loop transfer function between $\delta\gamma_i$ and δw_i of the ESJ system is given by

$$\frac{\delta\hat{\gamma}_i(s)}{\delta\hat{w}_i(s)}=\frac{H_i(s)g^i}{s^2a_{ii}-H_i(s)(f_1^i+f_2^is)+G_i(s)}. \tag{A.1}$$

The denominator polynomial function of (A.1) can, after some manipulation, be written as

$$D(s)=K_G\left(\frac{a_{ii}}{J_M}+N_i^2\right)\left[s^2+s\alpha_i(N_iK_B-f_2^i)-\alpha_if_1^i\right] + (\text{higher order terms in } s) \tag{A.2}$$

where

$$\alpha_i \triangleq \left(\frac{N_iK_T}{R_LJ_M}\right)\Big/\left(\frac{a_{ii}}{J_M}+N_i^2\right).$$

The two dominant poles of (A.1) are then given by the zeros of the polynomial

$$D'(s)=s^2+s\alpha_i(N_iK_B-f_2^i)-\alpha_if_1^i \tag{A.3}$$

where the gains f_1^i and f_2^i may, in principle, be chosen to yield any desired damping ratio and bandwidth for the dominant modes. Furthermore, from (A.1), the input gain g^i may also be chosen to satisfy steady-state requirements.

However, in addition to stability, most applications have constraint requirements on the control and state variables which must be met in order to avoid saturation problems. Suppose the control signal δv_i of (22) is not to exceed $\bar{v}$ volts in magnitude. We assume *a priori* that

$$|\delta\gamma_i(t)|\leqslant a, \quad |\delta\dot{\gamma}_i(t)|\leqslant b, \quad |\delta w_i(t)|\leqslant \bar{w}.$$

Let the input gain g_i be chosen such that at $\delta w_i=\bar{w}$, the steady-state deflection of $\delta\gamma_i$ is a; thus, by (A.1),

$$g^i=-\frac{af_1^i}{\bar{w}}. \tag{A.4}$$

Then it is easy to verify that $|\delta v_i|\leqslant\bar{v}$ provided the feedback gains satisfy the following inequality:

$$2a|f_1^i|+b|f_2^i|\leqslant\bar{v}. \tag{A.5}$$

Thus, for instance, a complete design for the feedback gains would be given by the following conditions:

$$\begin{aligned}\alpha_i(N_iK_B-f_2^i)&=2\xi\omega_n\\ -\alpha_if_1^i&=\omega_n^2\\ 2a|f_1^i|+b|f_2^i|&=\bar{v}\end{aligned} \tag{A.6}$$

where ξ and ω_n are, respectively, the damping ratio and natural frequency for the dominant modes of the closed-loop ESJ system. Since (A.6) contains three equations in only two design parameters, either ξ or ω_n, but not both, may be specified in order to yield a solution for the feedback gains.

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Robustness and Computational Aspects of Nonlinear Stochastic Estimators and Regulators

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***Abstract*—Robustness properties of nonlinear extended Kalman filters with constant gains and modeling errors are presented. Sufficient conditions for the nondivergence of state estimates generated by such nonlinear estimators are given. In addition, the overall robustness and stability properties of closed-loop stochastic regulators, based upon the linear-quadratic Gaussian design methodology using linearized dynamics, are presented; the sufficient conditions for closed-loop stability have a "separation-type" property.**

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I. Introduction

The substantial real-time computational burden imposed by the extended Kalman filter (EKF) and related suboptimal nonlinear estimators (cf. [1, ch. 6]) significantly limits the scope of applications for which these estimators are practical. The major portion of this computational burden results from calculations associated with propagating the error-covariance matrix, which in turn is used for real-time updating of the gain matrix acting on the filter residuals. When one considers the gross nature of the approximations that are routinely made in modeling the stochastic disturbances affecting a system and to a lesser extent in modeling the interplay between these disturbances and the system's nonlinearities, it seems somewhat surprising that so much real-time computational effort should be devoted to careful propagation of the model's error-covariance matrix. The empirical fact that the EKF performs well in many applications despite the gross nature of these routine approximations suggests that perhaps the record of successes enjoyed by the EKF is attributable to an intrinsic *robustness* against the effects of approximations introduced in the design of its residual-gain.

With a view towards designing nonlinear estimators with greatly reduced real-time computational requirements, we have been thusly motivated to examine the possibility of employing a *precomputed* approximation to the EKF residual-gain, thereby entirely *eliminating the enormous computational burden of real-time error-covariance propagation.* The principal implications of the results we have obtained in this connection are threefold.

First, we have found that the real-time propagation of error-covariance may actually be unnecessary. Specifically, our results imply that for many applications one can obtain satisfactory performance from a *constant-gain extended Kalman filter* (*CGEKF*), designed to be optimal for a stochastic linear time-invariant model crudely approximating the actual nonlinear system. Second, aided by the structural simplicity of the CGEKF, we have been able to apply modern input-output techniques of analysis to prove that *the CGEKF is intrinsically robust* against the effects of approximations introduced in the design of its residual-gain matrix. That is, we have proved that the CGEKF approach yields under certain conditions a *nondivergent* nonlinear estimator even when a relatively crude stochastic linear system model is used in designing the residual gain. Our nondivergence results take the form of analytically verifiable conditions which also can be used to test specific CGEKF designs for nondivergence, thereby reducing the engineer's dependence on Monte Carlo simulation for design validation. Moreover, the nature of these nondivergence conditions is such as to provide a basis for the constructive modification and improvement of CGEKF designs.Third, our results combine with the linear-quadratic-optimal-regulator robustness results of [2] and [3] in a fashion reminiscent of the *separation theorem* of estimation and control. We have proved that *any* nondivergent estimates— and this includes nondivergent CGEKF estimates— can be substituted for true values in a nonlinear feedback control system without inducing instability. This suggests a powerful new technique, based on linear-quadratic-Gaussian optimal feedback theory, for the synthesis of simplified dynamical output-feedback compensators for nonlinear regulator systems. The technique leads to a feedback compensator design consisting of a cascade of a CGEKF and an optimal constant linear-quadratic state-feedback (LQSF) gain matrix. We have proved that the inherent robustness of optimal linear-quadratic state-feedback against unmodeled nonlinearity [2], [3] combines with the intrinsic robustness of the CGEKF to assure that such feedback designs will be closed-loop stable even in systems with substantial nonlinearity.

The aforementioned CGEKF robustness, nondivergence, and regulator stability results are derived in the general context of the class of nonlinear estimators whose design is not necessarily based on statistical considerations—for example, designs intended to optimize structural simplicity or error-transient response, i.e., nonlinear observers (cf. [4]). This general class of nonlinear estimators include as a special case the CGEKF, which is suboptimally designed with respect to a statistical criterion. In the context of this broader class of suboptimal nonlinear estimators, our results provide analytically verifiable conditions which can be used to test nondivergence and to evaluate robustness against the effects of design approximations; though one cannot in general expect such designs to be as robust as the CGEKF. The CGEKF output-feedback separation-type property extends to this broader class of estimators, showing that nondivergent estimates can, unconditionally, be substituted true values in otherwise-stable feedback systems without ever causing instability.

II. Related Literature

The literature on the subject of robustness and computational considerations in nonlinear estimation is sparse and largely inconclusive. The discussion of nonlinear estimation in Schweppe [5, ch. 13] provides a good intuitive understanding of the trade-offs between computational requirements and residual-gain choice; though the possibility of a constant residual-gain is not explicitly considered. The idea of using a constant residual-gain for linear filtering is well known (cf. [1, pp. 238–242]), but the connection with nonlinear filtering has not been established. The practicality of avoiding real-time error-covariance propagation in nonlinear estimation by use of a precomputed, but estimate-dependent, residual-gain is demonstrated by the simulation results of Larson, Dressler, and Ratner [22]. References [4] and [6], which concern nonlinear estimator stability and error bounds, respectively, appear to be the most closely related to the present paper.

Gilman and Rhodes [6] suggest a procedure for synthesizing nonlinear estimators with a precomputable, but time-varying, residual-gain. Their estimator, like the traditional EKF, has the intuitively appealing structure of a *model-reference* estimator (cf. [5, p. 403]); that is, it consists of an internal model of the system dynamics with observations entering via a gain acting on the residual error between the system and model outputs. The distinguishing feature of the estimator suggested in [6] is that the residual-gain is chosen so as to minimize a certain upper bound on the mean-square estimate error. This procedure tends to ensure a robust design since, assuming the minimal value of the error-bound does not "blow-up," the estimator cannot diverge. A limitation of this design procedure is that the error-bound may be very loose for systems with substantial nonlinearity; so there is no assurance that the bound-minimizing residual-gain is a good choice. Also, there is no *a priori* guarantee that the resultant estimator will even be stable since the minimal error-bound may become arbitrarily large as time elapses.

Tarn and Rasis [4] have proposed a constant-gain model-reference-type nonlinear estimator which is a natural extension of Luenberger's observer for linear systems, having a design based solely on stability considerations. The results of [4] show that, given such a nonlinear observer design, if certain Lyapunov functions can be found, then one can conclude that

a) The estimator is nondivergent;

b) The estimator can be used for state reconstruction in a full-state feedback system without causing instability.

However, from an engineering standpoint they are nonconstructive: no design synthesis procedure is suggested; no method is proposed for constructing the Lyapunov functions required to test the stability of a design; no procedure is suggested for optimizing the estimate accuracy of the design. The CGEKF results presented in the present paper address all these deficienceis by providing a constructive procedure for synthesizing stable constant-gain model-reference estimator designs which are to a first approximation optimally accurate. Moreover, our results prove that, provided an estimator is nondivergent, it can be used for state reconstruction without ever causing instability, irrespective of the availability of Lyapunov functions.

III. Notation and Terminology

In this paper the input–output view of systems is taken, considering a *system* to be an interconnection of "black boxes" each representable by its input-output characteristics. As will become apparent, the input-output view provides a convenient and natural setting for the discussion and analysis of estimator robustness and divergence, as well as feedback system stability. In this section the pertinent terminology drawn from [7]–[11], and [20] is reviewed and the notion of estimator divergence is formalized.

An *operator* is a mapping of functions into functions—such as is defined by a "black box" which maps input time-functions into output time-functions. An operator is said to be *nonanticipative* if the value assumed by its output function at any time instant t_0 does not depend on the values assumed by its input function at times $t>t_0$. An operator is said to be *memoryless* or equivalently *nondynamical* if the instantaneous value of its output at time t_0 depends only on the value of its input at time t_0. A *dynamical* operator is an operator which is not necessarily nondynamical.

To facilitate the discussion, the various input and output functions considered in this paper are presumed to be imbedded in extensions of normed function spaces of the type

$$\mathfrak{M}_2(R_+,R^r) \triangleq \{z:R_+\to R^r;\ \|z\|<\infty\} \tag{3.1}$$

[11, p. 125] on which is defined the *norm*

$$\|z\| \triangleq \lim_{\tau\to\infty} \|z\|_\tau \tag{3.2}$$

where for all z, z_1, z_2

$$\|z\|_\tau \triangleq \langle z,z\rangle_\tau \tag{3.3}$$

$$\langle z_1,z_2\rangle_\tau \triangleq \frac{1}{\tau}\int_0^\tau z_1^T(t)z_2(t)\,dt \tag{3.4}$$

for all $\tau>0$.[1]

The quantity $\|z\|^2$ can be viewed as the "average power" in the function z; in fact, if z is generated by an ergodic random process (cf. [12, p. 327]), then $\|z\|^2$ is simply the expected value of $z^T(t)z(t)$.

Because the space $\mathfrak{M}_2$ may be unfamiliar to many readers, we briefly discuss its relation to the similar, but distinct, space $\mathfrak{L}_2$ which is more widely used in input–output system analysis. The feature that distinguishes $\mathfrak{M}_2$ from $\mathfrak{L}_2$ is the introduction of the "normalizing factor" $1/\tau$ into the inner product (3.4). Whereas the $\mathfrak{L}_2$-norm is appropriately viewed as a measure of the "total energy" of a function, the normalizing factor $1/\tau$ leads to the "average power" interpretation of the norm (3.2). The space $\mathfrak{M}_2$ is larger than $\mathfrak{L}_2$, every function in $\mathfrak{L}_2$ being included in the subspace of $\mathfrak{M}_2$ comprised of functions of zero norm.

The *gain* or *norm* of an operator $\mathfrak{F}$, denoted $g(\mathfrak{F})$ and $\|\mathfrak{F}\|$, respectively, are defined by

$$g(\mathfrak{F}) \triangleq \|\mathfrak{F}\| \triangleq \sup_{\substack{0<\|z\|_\tau<\infty\\ 0<\tau<\infty}} \frac{\|\mathfrak{F}z\|_\tau}{\|z\|_\tau}. \tag{3.5}$$

The *incremental gain* of $\mathfrak{F}$ is

$$\tilde{g}(\mathfrak{F}) \triangleq \sup_{\substack{0<\|z_1-z_2\|_\tau<\infty\\ 0<\tau<\infty}} \frac{\|\mathfrak{F}z_1-\mathfrak{F}z_2\|_\tau}{\|z_1-z_2\|_\tau}. \tag{3.6}$$

If $g(\mathfrak{F})<\infty$, $\mathfrak{F}$ is said to have *finite gain*. Likewise, if $\tilde{g}(\mathfrak{F})<\infty$, then $\mathfrak{F}$ is said to have *finite incremental gain*. The operator $\mathfrak{F}$ is *bounded* if inputs of finite norm produce outputs of finite norm; i.e., there exists a continuous, increasing function $\rho: R\to R$ such that $\|\mathfrak{F}z\|<\rho(\|z\|)$. A dynamical system is said to be *bounded* if the operator describing its input–output characteristics is bounded; the system is said to be *finite gain stable* if the operator has finite gain. An operator $\mathfrak{F}$ is said to be *strongly positive*, denoted $\mathfrak{F}>0$, if for some $\epsilon>0$ and all z,τ

$$\langle z,\mathfrak{F}z\rangle_\tau \geqslant \epsilon\|z\|_\tau^2. \tag{3.7}$$

If $\{\mathfrak{F}[x] \| x\in\mathfrak{X}\}$ is a collection of operators whose input–output relations are dependent upon the variable $x\in\mathfrak{X}$ and if, for some constant $\epsilon>0$ (which does not depend on x), (3.7) holds for all x, then we say that $\mathfrak{F}[x]$ is *uniformly strongly positive*; equivalently we write "uniformly for all x, $\mathfrak{F}[x]>0$." An operator $\mathfrak{F}$ is said to *positive*, denoted $\mathfrak{F}\geqslant 0$, if (3.7) holds with $\epsilon=0$.

[1]The extension of $\mathfrak{M}_2(R_+,R^r)$ consists of the set of all functions $z: R_+\to R^r$ such that for each $\tau\in R_+$ the quantity $\|z\|_\tau$ is finite. The extension includes functions like e^t which have infinite norm—cf. [20, part II].

The *Gateaux derivative* [8, p. 17] of the operator $\mathfrak{F}$ at the point z_0 is defined to be the bounded linear operator $\nabla\mathfrak{F}[z_0]$ having the property that for all z

$$\nabla\mathfrak{F}[z_0]z = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\mathfrak{F}(z_0+\epsilon z)-\mathfrak{F}z_0), \tag{3.8}$$

provided that such an operator $\nabla\mathfrak{F}[z_0]$ exists. When the derivative $\nabla\mathfrak{F}[z_0]$ exists, $\mathfrak{F}$ is said to be *Gateaux differentiable* at z_0. For example, if $\mathfrak{F}$ is memoryless, i.e., if $(\mathfrak{F}z)(t)\equiv f(z(t))$ for some $f: R^{r_1}\to R^{r_2}$, then $\nabla\mathfrak{F}[z_0]$ is simply the *Jacobian* matrix $(\partial f/\partial z)(z_0)$ (cf. [8, p. 19]). Alternatively, if $\mathfrak{F}$ is a linear operator then $\nabla\mathfrak{F}[z_0]=\mathfrak{F}$ for all z_0.

The relevance of the above terminology to estimation stems from the fact that for each control input function u, the error $e\triangleq\hat{x}-x$ of an estimator can be represented as the output of a u-dependent operator $\mathcal{E}[u]$ whose inputs are the system noise, say ξ, and measurement noise, say θ; i.e.,

$$e \triangleq \hat{x}-x=\mathcal{E}[u](\xi,\theta). \tag{3.9}$$

To formalize the notion of estimator divergence the following definitions are introduced: an estimator is *nondivergent* if its error operator is bounded uniformly in u, i.e., if there exists a continuous, increasing function $\rho: R\to R$ such that

$$\sup_u \|\mathcal{E}[u](\xi,\theta)\| \leqslant \rho(\|(\xi,\theta)\|); \tag{3.10}$$

it is *convergent* if $\rho(\cdot)\equiv 0$, it is *nondivergent with finite gain* if

$$\sup_u g(\mathcal{E}[u])<\infty. \tag{3.11}$$

Evidently, convergence implies nondivergence with finite gain which in turn implies nondivergence. These definitions can be loosely interpreted as follows: an estimator is nondivergent if mean-square bounded disturbances produce mean-square bounded estimate error; it is nondivergent with finite gain if the mean-square estimate error is proportional to the magnitude of the disturbances; it is convergent if the mean-square error always tends to zero. An estimator that is not nondivergent is said to be *divergent*.

IV. Problem Formulation

We consider the problem of estimation for the nonlinear system

$$\left.\begin{aligned} \frac{d}{dt}x &= \mathcal{Q}[w]x+\mathfrak{B}[w]u+\xi; \quad x(0)=0\\ y &= \mathcal{C}[w]x+\theta \end{aligned}\right\} \tag{4.1}$$

where w is a vector of functions including y,u,t as well as all other *known* or *observed* functions (e.g., estimates $\hat{x}$ of x generated from observations and known exogenous inputs to the system);

$\mathcal{Q}[w]$, $\mathfrak{B}[w]$, $\mathcal{C}[w]$ are (for each w) nonanticipative, Gateaux differentiable, *dynamical nonlinear operators* with *finite incremental gain*;

$\xi\in\mathfrak{M}_2(R_+,R^n)$, $\theta\in\mathfrak{M}_2(R_+,R^p)$ are disturbance input functions;

y is an R^p-valued observed output function;

u is an R^m-valued known control input function;

x is an R^n-valued function which is to be estimated based on knowledge of y and u.

We refrain from specifying the statistical properties (e.g., the mean and the covariance) of $\xi(t)$ and $\theta(t)$ at this point as these have no bearing on our general results in Section V. However, the statistical properties of $\xi(t)$ and $\theta(t)$ play a role in CGEKF design which we address in subsequent sections.

As a candidate for estimator for the system (4.1) we consider the model-reference estimator

$$\left.\begin{aligned} \frac{d}{dt}\hat{x} &= \mathcal{Q}[w]\hat{x}+\mathfrak{B}[w]u-H[w](\hat{y}-y)\\ \hat{y} &= \mathcal{C}[w]\hat{x} \end{aligned}\right\} \tag{4.2}$$

where $H[w]$ is a matrix of appropriate dimensions whose entries depend on w. When $\mathcal{Q}[w]$, $\mathcal{C}[w]$, and $H[w]$ are independent of w and when $\mathcal{Q}[w]$

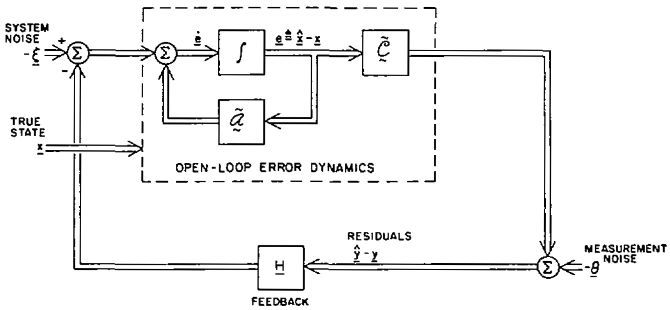

Fig. 1. Feedback representation of the error dynamics of the nonlinear observer.

and $\mathcal{C}[w]$ are nondynamical, then (4.2) is identical in structure to the so-called "observer for nonlinear stochastic systems" proposed by Tarn and Rasis [4]; consequently, we refer to the structure (4.2) as a *nonlinear observer*. An extended Kalman filter is a special type of nonlinear observer in which the gain $H[w]$ is suboptimally designed based on statistical considerations.

A useful method for describing the dynamical evolution of the nonlinear observer's error

$$e \triangleq \hat{x} - x, \tag{4.3}$$

is by the feedback equations (see Fig. 1)

$$\left.\begin{aligned} \frac{d}{dt}e &= \tilde{\mathcal{A}}[w,x]e + v; \qquad e(0)=0 \\ r &= \tilde{\mathcal{C}}[w,x]e - \theta \end{aligned}\right\} \tag{4.4}$$

$$v = -H[w]r - \xi \tag{4.5}$$

where

$$r \triangleq \hat{y} - y \tag{4.6}$$

and $\tilde{\mathcal{A}}[w,x]$ and $\tilde{\mathcal{C}}[w,x]$ are dynamical nonlinear operators defined by

$$\tilde{\mathcal{A}}[w,x]z = \mathcal{A}[w](x+z) - \mathcal{A}[w]x \tag{4.7}$$

$$\tilde{\mathcal{C}}[w,x]z = \mathcal{C}[w](x+z) - \mathcal{C}[w]z \tag{4.8}$$

for all $z \in \mathfrak{M}_2(R_+, R^n)$. From this feedback representation of the error dynamics of the nonlinear observer (4.2), it is immediately apparent that the problem of choosing the residual-gain $H[\cdot]$ so as to make the estimator nondivergent is identical to the problem of choosing a stabilizing feedback for the system (4.4).

In order to facilitate the selection of a suitable residual-gain $H[\cdot]$, we assume that (4.4) describing the "open-loop error-dynamics" admits the nominal linearization

$$\begin{aligned} \frac{d}{dt}e &= A[w]e + v; \qquad e(0)=0 \\ r &= C[w]e \end{aligned} \tag{4.9}$$

where $A[w]$ and $C[w]$ are matrices of appropriate dimensions whose entries, in general, depend nondynamically on w.[2] In the case where A and C are chosen to be constant (i.e., independent of w) the problem is reduced to a time-invariant linear estimation problem for which several methods are available for choosing $H[\cdot]$; e.g., pole assignment [13, Sec. 7.4] or Kalman filtering [14, ch. 7]. Also, the simplest estimator structure results when A, C, and hence H are chosen to be independent of w.

V. Nonlinear Observer Results

We now state two basic theorems concerning the nonlinear observer (4.2). The first result, Theorem 1, states that substitution of estimates generated by a nondivergent nonlinear observer for true values in an otherwise-stable feedback control system can never destabilize the closed-loop system. This result has obvious implications regarding the utility of nonlinear observers for state reconstruction in nonlinear optimal and suboptimal feedback control systems. The second result, Theorem 2, gives sufficient conditions for a nonlinear observer to be nondivergent. The proofs are in the Appendix.

Theorem 1: Let $\mathcal{G}$ be a nonanticipative nonlinear dynamical operator with finite incremental gain. Suppose that the system (4.1) is closed-loop bounded (finite gain stable) with feedback $u = \mathcal{G}x$. Then the system (4.1) with feedback $u = \mathcal{G}\hat{x}$ will also be closed-loop bounded (finite gain stable), provided that the estimate $\hat{x}$ is nondivergent (with finite gain).

Theorem 2: Let the w-dependent matrix $S[w]$ and the *constant* matrix P be symmetric uniformly positive-definite solutions of the w-dependent Lyapunov equation

$$(A[w] - H[w]C[w])P + P(A[w] - H[w]C[w])^T + S[w] = 0. \tag{5.1}$$

If uniformly for all x, w

$$\{A[w] - \nabla(\mathcal{A}[w])[x] - H[w](C[w]) - \nabla(\mathcal{C}[w])[x]\}P + \frac{1}{2}S[w] > 0, \tag{5.2}$$

then the nonlinear observer (4.2) is nondivergent with finite gain.

The condition (5.1) is not a severe restriction; it specifies, in essence, that the matrix P must be chosen such that $x^T(t)Px(t)$ is a positive-definite Lyapunov function ensuring closed-loop stability for the ideal situation in which the linearization (4.9) exactly models the actual error dynamics. For example, when A, C, and H are constant matrices and H stabilizes the error-dynamics feedback system (4.4), (4.5), a constant matrix P satisfying (5.1) can be readily found by simply picking *any* positive-definite constant matrix S and solving (5.1) for the (unique!) positive definite solution P satisfying (5.1) [13, p. 341].

The interesting part of Theorem 2 is the condition (5.2). It characterizes a class of nonlinearities for which the nonlinear observer (4.2) is assured of being nondivergent. An important feature of Theorem 2 is the *form* of conditions (5.2): it is expressed in terms of the *deviation* of the system (4.4) from the linearization (4.9) used in selecting the residual-gain. When the deviation is zero (i.e., $(\mathcal{A}, \mathcal{C}) \equiv (A, C)$) then the condition (5.2) is always satisfied since S is positive definite.

The question naturally arises "How difficult is it to verify condition (5.2)?" The fact that the left-hand side of (5.2) is linear in $\mathcal{A}$ and $\mathcal{C}$ and the fact that a positively weighted sum of positive operators is positive, make (5.2) much easier to verify than might be apparent at first inspection. For example, suppose $\mathcal{A}$, $\mathcal{C}$, A, C, and H are nondynamical and (for simplicity) independent of w. If there are constants $c_{ij}^{(l)}$, $a_{jk}^{(l)}$ ($l=1,2$; $i=1,\cdots,p$; $j,k=1,\cdots,n$) such that for all $x \in R^n$

$$0 \geq c_{ij}^{(1)} \leq [C - \nabla\mathcal{C}[x]]_{ij} \leq c_{ij}^{(2)} \geq 0 \tag{5.3}$$

$$0 \geq a_{jk}^{(1)} \leq [A - \nabla\mathcal{A}[x]]_{jk} \leq a_{jk}^{(2)} \geq 0 \tag{5.4}$$

(where $[M]_{ij}$ denotes the ijth element of the matrix M), then one may readily verify that sufficient conditions for (5.2) to hold are

$$c_{ij}^{(l)}\left(Pe_je_i^TH^T + He_ie_j^TP\right) + S > 0 \tag{5.5}$$

$$a_{jk}^{(l)}\left(Pe_ke_j^T + e_je_k^TP\right) + S > 0 \tag{5.6}$$

(where e_i denotes the ith standard basis vector, i.e., the vector whose elements are zero except the ith which is a one). To verify conditions (5.5) and (5.6) requires that one check the positive definiteness of as many $n \times n$-matrices as there are nonzero elements in the set

[2] For example, if $(\mathcal{A}[w]x)(t) \equiv f(x(t), u(t))$ and $(\mathcal{C}[w]x)(t) \equiv h(x(t))$ for some functions $f: R^n \to R^n$ and $g: R^n \times R^m \to R^p$, then it might be reasonable to choose

$$A[w](t) = \frac{\partial f}{\partial x}(x(t), u(t))$$

$$C[w](t) = \frac{\partial h}{\partial x}(x(t));$$

this is the choice traditionally advocated for extended Kalman filter design.

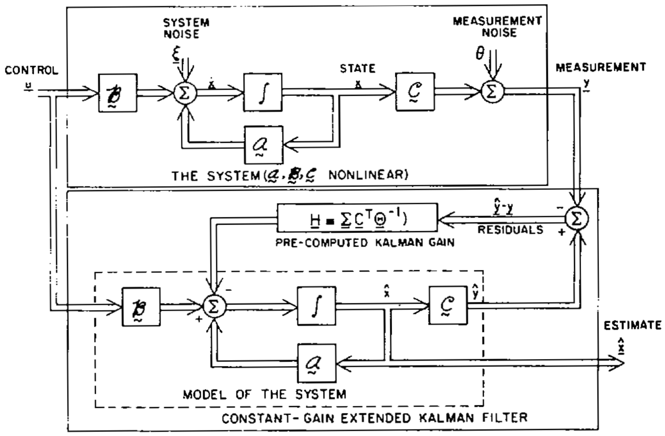

Fig. 2. The system with constant-gain extended Kalman filter (CGEKF).

$$\left\{ c_{ij}^{(l)}, a_{jk}^{(l)} \mid l=1,2;\ i=1,\cdots,p;\ j,k=1,\cdots,n \right\}$$

(which can be done, for example, by checking that the principal leading minors of each matrix are positive [13, p. 341]). So, if the nonlinear system (4.4) is identical to the linearization (4.9) except for N memoryless nonlinearities, then one need only check the positive definiteness of at most $2N$ $n \times n$-matrices to verify (5.2).

VI. The Constant-Gain Extended Kalman Filter (CGEKF)

Intuitively, it is clear that if the linearization (4.9) is sufficiently faithful to the nonlinear system (4.4), then the error response of the nonlinear observer (4.2) will be close to the error response one would get in the ideal situation in which the linearization is exact. This intuition is validated by the error-bounding results of [6] and [15]. Consequently, if the disturbances $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ are reasonably well approximated by zero-mean white noise, then it is reasonable to expect that a good suboptimal minimum variance estimator can be obtained by choosing the residual-gain $\boldsymbol{H}$ to be the minimum-variance-optimal gain for the linearized system (4.9), i.e., the Kalman filter gain [14, p. 214]

$$\boldsymbol{H}[\boldsymbol{w}] = \boldsymbol{\Sigma}[\boldsymbol{w}] \boldsymbol{C}^T[\boldsymbol{w}] \boldsymbol{\Theta}^{-1}[\boldsymbol{w}] \tag{6.1}$$

where $\boldsymbol{\Sigma}[\boldsymbol{w}] = \boldsymbol{\Sigma}^T[\boldsymbol{w}] > \boldsymbol{0}$ satisfies the Riccati equation[3]

$$\boldsymbol{0} = \boldsymbol{\Sigma}[\boldsymbol{w}]\boldsymbol{A}^T[\boldsymbol{w}] + \boldsymbol{A}[\boldsymbol{w}]\boldsymbol{\Sigma}[\boldsymbol{w}] - \boldsymbol{\Sigma}[\boldsymbol{w}]\boldsymbol{C}^T[\boldsymbol{w}]\boldsymbol{\Theta}^{-1}[\boldsymbol{w}]\boldsymbol{C}[\boldsymbol{w}]\boldsymbol{\Sigma}[\boldsymbol{w}] + \boldsymbol{\Xi}[\boldsymbol{w}] \tag{6.2}$$

and $\boldsymbol{\Xi}[\boldsymbol{w}]$ and $\boldsymbol{\Theta}[\boldsymbol{w}]$ are ($\boldsymbol{w}$-dependent) positive-definite covariance matrices of the disturbances $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$, respectively. When $\boldsymbol{H}$ is constant (i.e., independent of $\boldsymbol{w}$), the resultant estimator is the constant-gain extended Kalman filter (CGEKF) depicted in Fig. 2.

A surprising and important consequence of the CGEKF approach to nonlinear observer design is that, in addition to yielding a suboptimally accurate estimator design, *the CGEKF design procedure is inherently robust* in the sense that even a crude linearization (4.9) will suffice for residual-gain design. The CGEKF design procedure automatically ensures that the deviation from the design linearization admissible under the conditions of Theorem 2 can be quite large. The extent of this robustness is quantified in the following result.

Theorem 3 (CGEKF Robustness): If $\boldsymbol{\Sigma}$ is independent of $\boldsymbol{w}$ and if uniformly for all $\boldsymbol{x} \in \mathfrak{M}_2(R_+, R^n)$ and all $\boldsymbol{w}$

$$\{\boldsymbol{A}[\boldsymbol{w}] - \nabla(\mathcal{G}[\boldsymbol{w}])[\boldsymbol{x}] - \boldsymbol{H}[\boldsymbol{w}](\boldsymbol{C}[\boldsymbol{w}] - \nabla(\mathcal{C}[\boldsymbol{w}])[\boldsymbol{x}])\}\boldsymbol{\Sigma}$$
$$+ \frac{1}{2}(\boldsymbol{\Xi}[\boldsymbol{w}] + \boldsymbol{\Sigma}\boldsymbol{C}^T[\boldsymbol{w}]\boldsymbol{\Theta}^{-1}[\boldsymbol{w}]\boldsymbol{C}[\boldsymbol{w}]\boldsymbol{\Sigma}) > \boldsymbol{0}, \tag{6.3}$$

then the CGEKF is nondivergent with finite gain.[4]

[3]We assume that the required controllability and observability conditions are satisfied so that there is a unique positive definite solution of (6.2) (cf. [14, pp. 234–243]).

[4]Since $\boldsymbol{\Sigma}$ is determined by (6.2), a sufficient condition for $\boldsymbol{\Sigma}$ to be independent of $\boldsymbol{w}$ is that (6.2) be independent of $\boldsymbol{w}$, i.e., that A, C, Ξ, and Θ be independent of $\boldsymbol{w}$.

Proof: Let

$$\left.\begin{aligned} \boldsymbol{S}[\boldsymbol{w}] &= \boldsymbol{\Xi}[\boldsymbol{w}] + \boldsymbol{\Sigma}\boldsymbol{C}^T[\boldsymbol{w}]\boldsymbol{\Theta}^{-1}[\boldsymbol{w}]\boldsymbol{C}[\boldsymbol{w}]\boldsymbol{\Sigma} \\ \boldsymbol{P} &= \boldsymbol{\Sigma}. \end{aligned}\right\} \tag{6.4}$$

Then (6.2) and (6.3) ensure that (5.1) and (5.2), respectively, are satisfied. The result follows from Theorem 2.

Example: To illustrate the application of Theorem 3, we consider the problem of designing an estimator for the nonlinear system

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t)) + g(u(t)) + \xi(t) \\ y(t) &= h(x(t)) + \theta(t) \end{aligned}\right\} \tag{6.5}$$

where

$$y(t) \in R^1$$
$$x(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in R^2$$
$$\xi(t) \in R^2$$
$$f(x) \equiv \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \triangleq \begin{bmatrix} -\frac{1}{2}(x_1 + x_1^3) - x_2 \\ x_1 \end{bmatrix} \in R^2$$
$$g(u) = \begin{bmatrix} \operatorname{sgn}(u) \\ 0 \end{bmatrix} \in R^2$$
$$h(x) = x_2 \in R^1.$$

If we suppose that $\xi(t)$ and $\theta(t)$ are Gaussian white noise with covariance matrices Ξ and Θ, respectively, and if we employ the linear model

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + \xi \\ y &= Cx + \theta \end{aligned}\right\} \tag{6.6}$$

with

$$A = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$C = [0 \quad 1]$$
$$\Xi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\Theta = 1,$$

then solving (6.1) and (6.2) for Σ and H yields

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.7}$$

$$H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.8}$$

The resultant CGEKF estimator is

$$\dot{\hat{x}} = f(\hat{x}) + g(u) - H(h(\hat{x}) - y). \tag{6.9}$$

For every function $x: R_+ \to R^n$, define the operator $\mathfrak{F}[x]: \mathfrak{M}_2(R_+, R^2) \to \mathfrak{M}_2(R_+, R^2)$ by

$$[\mathfrak{F}[x]\eta](t) = \left[A - \frac{\partial f}{\partial x}(x(t)) - H\left(C - \frac{\partial h}{\partial x}(x(t))\right)\right]\Sigma\eta(t) \tag{6.10}$$

for all $\eta \in \mathfrak{M}_2(R_+, R^2)$ and all $t \in R_+$. By Theorem 3, a sufficient condition for the CGEKF (6.9) to be nondivergent is that $\mathfrak{F}[x]$ be uniformly strongly positive for all x. This requires that there exist some positive constant ϵ such that for every $x: R_+ \to R^2$ and every $\tau \in R_+$

$$\langle \eta, \mathfrak{F}[x]\eta \rangle_\tau - \epsilon \|\eta\|_\tau^2$$
$$\equiv \frac{1}{\tau}\int_0^\tau \eta^T(t)\left(\left[A - \frac{\partial f}{\partial x}(x(t)) - H\left(C - \frac{\partial h}{\partial x}(x(t))\right)\right]\Sigma\right.$$

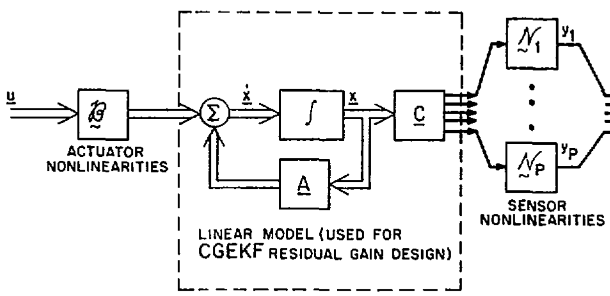

Fig. 3. System with all nonlinearity lumped in actuators and sensors.

$$+\frac{1}{2}(\Xi+\Sigma C^T\Theta^{-1}C\Sigma)-\epsilon\begin{bmatrix}1&0\\0&1\end{bmatrix}\Big]\Big)\eta(t)\,dt>0. \tag{6.11}$$

This will be the case if and only if the x-dependent matrix

$$\Big[A-\frac{\partial f}{\partial x}(x)-H\Big(C-\frac{\partial h}{\partial x}(x)\Big)\Big]\Sigma+\frac{1}{2}(\Xi+\Sigma C^T\Theta^{-1}C\Sigma)$$
$$=\begin{bmatrix}\left(\frac{3}{2}x_1^2+\frac{1}{2}\right)&0\\0&1\end{bmatrix} \tag{6.12}$$

is uniformly positive definite for all $x\in R^2$. Clearly, the matrix is uniformly positive definite; so the CGEKF (6.9) is assured of being nondivergent with finite-gain. (End of Example)

To fully appreciate the implications of Theorem 3 with regard to the robustness of the CGEKF design procedure, it is instructive to consider the situation in which for all x

$$\underset{\sim}{\mathcal{A}}[x]=A \tag{6.13}$$

$$\underset{\sim}{\mathcal{C}}[x]=[\mathrm{diag}(\mathcal{N}_1,\cdots,\mathcal{N}_p)]C \tag{6.14}$$

so that all the differences between the open-loop error dynamics system (4.4) and the design linearization (4.9) are lumped into the p dynamical nonlinearities, $\mathcal{N}_i$ $(i=1,\cdots,p)$, which are in series with the system outputs. This is equivalent to all nonlinearity in the system (4.1) being lumped in the actuators and sensors (see Fig. 3). It is emphasized that this does not mean that we are restricting our attention to systems with only actuator and sensor nonlinearity; rather, we are merely stipulating that the actual system's open-loop error dynamics (4.4) have the same input-output behavior as such a system.

For simplicity, we further assume θ is of the form

$$\Theta=\mathrm{diag}(\theta_{11},\theta_{22},\cdots,\theta_{pp}). \tag{6.15}$$

With (6.13)–(6.15) satisfied, the nondivergence condition (6.3) of Theorem 3 reduces to

$$\Sigma C^T\mathrm{diag}\Big[\theta_{11}^{-1}\Big(\nabla\mathcal{N}_1[x]-\frac{1}{2}\Big),\cdots,\theta_{pp}^{-1}\Big(\nabla\mathcal{N}_p[x]-\frac{1}{2}\Big)\Big]C\Sigma+\frac{1}{2}\Xi>0, \tag{6.16}$$

which is satisfied if

$$\nabla\mathcal{N}_i[x]\geq\frac{1}{2}\qquad(i=1,\cdots,p). \tag{6.17}$$

The condition (6.17) establishes a "lower bound" the inherent robustness of the CGEKF design procedure, i.e., every CGEKF design can tolerate *at least* nonlinearities satisfying (6.17). One can interpret this inherent robustness in terms of the *gain and phase margin* of the feedback representation (cf. Fig. 1) of the CGEKF error dynamics as follows: Suppose that the $\mathcal{N}_i$ $(i=1,\cdots,p)$ are stable linear dynamical elements with respective to transfer functions $L_i(s)$ $(i=1,\cdots,p)$. Then, condition (6.17) becomes (as a consequence of Parseval's theorem)

$$\mathrm{Re}[L_i(j\omega)]\geq\frac{1}{2}\qquad(i=1,\cdots,p); \tag{6.18}$$

i.e., the Nyquist locus of each $L_i(j\omega)$ must lie to the right of the vertical line in the complex plane passing through the point $1/2+j0$. For example, if $L_i(s)$ $(i=1,\cdots,p)$ are nondynamical linear gains, i.e., $L_i(j\omega)=k$, then (6.18) becomes

$$k\geq\frac{1}{2}. \tag{6.19}$$

Alternatively, if

$$L_i(s)=e^{j\phi_i}\qquad(i=1,\cdots,p)$$

corresponding to a pure phase shift of angle ϕ_i $(i=1,\cdots,p)$ in the p respective output channels of the open-loop error dynamics system, then condition (6.18) becomes

$$|\phi_i|\leq 60°. \tag{6.20}$$

One can interpret the conditions (6.19) and (6.20) as saying *the CGEKF design procedure leads to an infinite gain margin, at least* 50 *percent gain reduction tolerance, and at least* $\pm 60°$ *phase margin in each output channel* of the error dynamics feedback system (Fig. 1)—the margins being relative to the ideal situation in which the linearization (4.9) is exact. Engineers experienced in classical servomechanism design will recognize that these minimal stability margins are actually quite large, ensuring that the nonlinear observer error dynamics feedback system of Fig. 1 will be stable despite substantial differences between the design linearization (4.9) and the system (4.4). Consequently, the CGEKF design procedure is assured of yielding a nondivergent nonlinear observer design for systems with a good deal of nonlinearity.

This surprisingly large robustness of the CGEKF design procedure is mathematically dual to the robustness of linear-quadratic state-feedback regulators reported in [2] and [3], wherein full-state-feedback linear optimal regulators are shown to have infinite gain margin, 50 percent gain reduction tolerance, and $\pm 60°$ phase margin in each *input* channel. This duality is a consequence of the symmetry between the equations governing the regulation error of linear optimal regulators and the equations governing the estimate error of the CGEKF (cf. [2, (B.1) and (4.3)] versus (4.4) and (6.2) here).

VII. Practical CGEKF Synthesis

The results of the preceding section provide a basis for computer-aided-design of practical, nondivergent CGEKF estimators. The following procedure shows how these results might be employed for this purpose.

Step 1: Pick constant values for A, C, Ξ and Θ. The values of A and C should be initially chosen to reflect as closely as possible the derivatives $\nabla\mathcal{A}[x]$ and $\nabla\mathcal{C}[x]$, respectively, i.e., so that $\|A-\nabla\mathcal{A}[x]\|$ and $\|C-\nabla\mathcal{C}[x]\|$ are small, at least for those values of x and w which are most probable—statistical linearization methods (cf. [16, ch. 7]) may be helpful in this regard. The matrices Θ and Ξ should be initially chosen to reflect the covariances of the disturbances θ and ξ, respectively. If the input-output relations of the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are not precisely known, then the designer may wish to consider compensating for this using state-augmentation following the spirit of [17] and [18] in order to reduce bias errors.

Step 2: Compute Σ and H from (6.1) and (6.2). This can be done with the aid of a digital computer using available software for solving the Riccati equation.

Step 3: Test the resultant CGEKF design for nondivergence. This can be done any of the following ways:

1) By checking the conditions of Theorem 3;
2) By direct digital Monte Carlo simulation;
3) By approximate describing-function simulation [1, Sec. 6.4].

If the estimator is nondivergent, go to Step 5; otherwise, proceed to Step 4.

Step 4: Take the divergent CGEKF and, assisted by a computer, determine the values of x for which the condition (6.3) is not satisfied. Modify the matrices A and C so as to reduce the magnitude $\|A-\nabla\mathcal{A}[x]\|$ and $\|C-\nabla\mathcal{C}[x]\|$ at these values of x. If necessary, adjust the Ξ and Θ matrices. Return to Step 2.

Step 5: Check the nondivergent CGEKF for satisfactory performance, i.e., for acceptable error statistics. This can be done using one or more of the following approaches:

1) By direct digital Monte Carlo simulation;
2) By approximate describing-function simulation [1, Sec. 6.4];
3) By using the error-bounding results in [6] and [15].

If performance is acceptable, stop. Otherwise, further adjust the values of the constant matrices $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Theta}$ as in Step 4 and return to Step 2.

(End of Procedure)

For systems that are not "too nonlinear" this procedure can be expected to converge rapidly to an acceptable CGEKF design. However, for highly nonlinear systems, the procedure may not lead easily to a satisfactory design, even when such a design exists. A noteworthy limitation of the procedure is that no explicit method is provided for selecting the "best" modifications of $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Theta}$ as required in Step 5. Even in cases where a nondivergent CGEKF estimator is not practical it may be possible to exploit Theorem 3 to construct a gain-scheduled CGEKF estimator which, if properly initialized and if not subjected to excessively large disturbances, has satisfactory performance. This is accomplished by using estimate-dependent matrices $\boldsymbol{A}[\hat{x}]$, $\boldsymbol{C}[\hat{x}]$, $\boldsymbol{\Theta}[\hat{x}]$, and $\boldsymbol{\Xi}[\hat{x}]$ leading to estimate-dependent $\boldsymbol{\Sigma}[\hat{x}]$ and $\boldsymbol{H}[\hat{x}]$. That is, $\boldsymbol{H}[\hat{x}]$ is "scheduled" according to $\hat{x}$. The on-line computations for such an estimator would be necessarily more burdensome than for a CGEKF; but, provided a simple enough gain scheduling algorithm is employed, this burden would be substantially less than that of an EKF for which on-line error-covariance propagation is required. For each fixed value of $\hat{x}$—i.e., $\hat{x}(t) \equiv x_0 = \text{constant}$—condition (6.3) of Theorem 3 defines a subset of R^n having the property that provided the true state trajectory remains for all future time within that set, the CGEKF with constant residual gain matrix $\boldsymbol{H}(x_0)$ is stable and nondivergent. Thus, Theorem 3 serves to determine the number of location of fixed values—viz. $x = x_0$ —at which it is necessary to compute values for $\boldsymbol{H}$ in order to cover the entire reachable state space with stabilizing constant residual gain matrices. This information is useful in assessing whether a gain scheduled design is practical and how complicated the gain scheduling algorithm must be. It should be emphasized, however, that the results of Theorems 2 and 3 do not apply rigorously in the case of an estimate-dependent $\boldsymbol{\Sigma}$. Consequently, such *gain-scheduled* estimators may require careful initialization and may not be able to recover from large disturbances without reinitialization, much like the traditional EKF which, in general, has similar limitations.

VIII. Suboptimal Nonlinear Output-Feedback Controllers

The CGEKF results of the present paper combine with the results of [2] and [3] on the nonlinearity tolerance of linear-quadratic state-feedback (LQSF) control laws to suggest a simple, practical *nonlinear* extension of the celebrated linear-quadratic Gaussian optimal output-feedback control design technique. The idea is to cascade a CGEKF estimator with a constant LQSF gain matrix, both optimally designed for the time-invariant nominal linearization of the system (4.1)

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + \xi; \quad x(0) = 0 \\ y &= Cx + \theta \end{aligned}\right\} \tag{8.1}$$

with performance index

$$J(x,u) \triangleq E\left[\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} x^T(t) Q x(t) + u^T(t) R u(t)\, dt \right] \tag{8.2}$$

where

$\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ are zero-mean white Gaussian with respective covariance matrices $\boldsymbol{\Xi}$ and $\boldsymbol{\Theta}$;

$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are matrices of appropriate dimensions;

$\boldsymbol{R}$, $\boldsymbol{Q}$ are positive definite weighting matrices of appropriate dimensions. *It is assumed for simplicity* that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ *are not w-dependent*. For the linearization (8.1), the optimal Kalman filter residual gain is given by (6.1) and (6.2) and the optimal LQSF feedback is given by

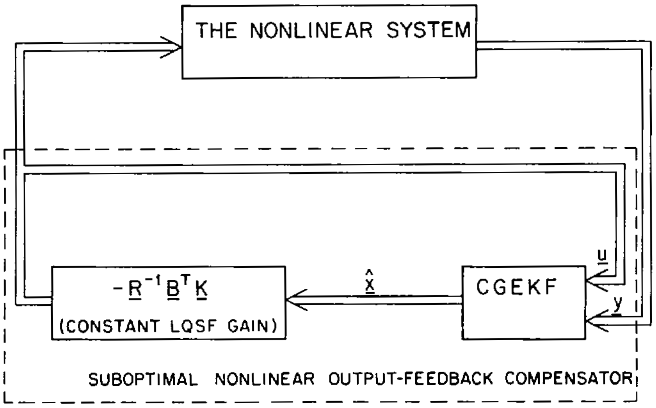

Fig. 4. Suboptimal nonlinear output-feedback controller.

$$u = -R^{-1}B^T K x \tag{8.3}$$

where $\boldsymbol{K} = \boldsymbol{K}^T \geqslant \boldsymbol{0}$ satisfies the time-invariant Ricatti equation

$$0 = KA + A^T K - KBR^{-1}B^T K + Q. \tag{8.4}$$

Cascading the CGEKF with the feedback (8.3) leads to the suboptimal nonlinear output-feedback control law (see Fig. 4)

$$\begin{aligned} u &= -R^{-1}B^T K \hat{x} \\ \frac{d}{dt}\hat{x} &= \mathcal{A}\hat{x} + \mathcal{B}u - \Sigma C^T \Theta^{-1}(\hat{y} - y) \\ \hat{y} &= \mathcal{C}x. \end{aligned} \tag{8.5}$$

This approach to suboptimal nonlinear output-feedback control design is similar in spirit to the approach outlined in [19], wherein an extended Kalman filter is cascaded with a *time-varying* suboptimal feedback gain; however the precomputed constant gains in the control law (8.5) make it drastically simpler to implement from the standpoint of real-time computational burden. The remarkable robustness of the CGEKF design procedure and of LQSF control designs [2], [3] assure that this approach will produce a stabilizing feedback control law for systems with even substantial nonlinearity. The extent of this robustness is quantified in the following result.

Theorem 4 (nonlinear output-feedback robustness): If uniformly for all x

$$[A - \nabla\mathcal{A}[x] + (-\Sigma C^T \theta^{-1})(C - \nabla\mathcal{C}[x])]\Sigma + \frac{1}{2}(\Xi + \Sigma C^T \Theta^{-1} C \Sigma) > 0 \tag{8.6}$$

and if

$$K[A - \mathcal{A} + (B - \mathcal{B})(-R^{-1}B^T K)] + \frac{1}{2}(Q + KBR^{-1}B^T K) > 0, \tag{8.7}$$

then the system (4.1) with output-feedback (8.5) (as is depicted in Fig. 4) is finite gain stable.

Proof: This result is a direct consequence of [2, theorem B.1] and of Theorems 1 and 3 of this paper: applying Theorem 3, condition (8.6) ensures that the CGEKF is nondivergent with finite gain; applying [2, theorem B.1], condition (8.7) ensures that the system (4.1) with full-state feedback (8.3) is stable with finite gain; the result follows from Theorem 1.

IX. Conclusions

Efforts to find methods for reducing the real-time computational burden of the extended Kalman filter have led us to consider the possibility of a constant-gain extended Kalman filter (CGEKF), designed to be optimal for a constant linear approximation of the actual nonlinear system. Since the residual-gain for a CGEKF estimator is constant and precomputable, the enormous real-time computational burden of error-covariance propagation and residual-gain updating is eliminated, drastically reducing real-time computational requirements. Because in many applications the linearization and disturbance modeling approximations made in CGEKF design may be only slightly cruder than the gross approximations that are made in EKF design, it is expected that the error-performance of CGEKF designs may actually be competitive with traditional EKF designs.

By representing a nonlinear estimator as a servomechanism in which error is the output to be regulated, we have been able to apply modern input-output techniques of analysis to generate results explicitly characterizing the robustness of CGEKF estimators—and, more generally, estimators having the structure of the nonlinear observer (4.2)—against the effects of approximations introduced in designing the residual gain. This result provides conditions characterizing the amount of *deviation* of the constant linear residual-gain design model from the actual nonlinear system that can be tolerated by specific CGEKF designs without the possibility of divergent estimates. Additionally, we have found that every CGEKF has a certain intrinsic robustness against divergence which is interpretable as an infinite gain margin and at least a $\pm 60°$ phase margin in each output channel of the associated error-dynamics feedback system. The synthesis of practical CGEKF designs has been discussed and it has been shown that the CGEKF nondivergence conditions can be exploited to constructively modify and improve CGEKF designs.

Of fundamental significance is the "separation" result (Theorem 1) which shows that nondivergent estimates can, unconditionally, be substituted for true values in feedback control systems without inducing instability. This provides theoretical justification for the use of extended Kalman filters (including CGEKF's) and other types of nonlinear observers for state reconstruction in nonlinear feedback control systems.

A new method, based on linear-quadratic-Gaussian optimal feedback theory, has been proposed for the synthesis of suboptimal output-feedback control laws for nonlinear systems. The method leads to a simply-structured nonlinear dynamical feedback law that is drastically simpler to implement than suboptimal linear-quadratic-Gaussian nonlinear feedback controllers incorporating a time-varying gain and a traditional EKF (cf. [19]). The feedback law decomposes naturally into an LQSF gain matrix and a CGEKF estimator in a fashion reminiscent of the way the separation theorem of estimation and control leads to a similar decomposition in linear problems. It has been shown that the inherent robustness of the CGEKF design procedure and of linear-quadratic state-feedback combine to assure that this design approach will lead to a stable feedback law for systems with substantial nonlinearity.

A limitation of the scope of the results of this paper is that they concern primarily such "coarse" measures of system performance as stability and nondivergence. "Finer" measures such as error-covariance are not explicitly addressed; though *the CGEKF design procedure tends to ensure that error-covariance is approximately minimized* when the linear design model matrices (A, C) closely approximate the nonlinear operators $(\mathcal{Q}, \mathcal{C})$. This relative de-emphasis of error covariance is partially justified by the fact that the covariances of the disturbances ξ and θ are seldom precisely known in practice; so it is seldom practical to precisely analyze the statistical properties of an estimate. Nondivergence and stability properties—properties that are prerequisite to a bounded error covariance—can be analytically assessed independently of statistical considerations, as we have demonstrated.

Appendix

In this Appendix the results of Zames [7] (as elaborated upon in [9], [20]) are used to prove Theorems 1 and 2. For compactness of notation *the argument w associated with the various operators and matrices has been suppressed.* We begin by introducing a definition.

Definition: Let $\mathcal{F}: \mathfrak{X} \to \mathfrak{Y}$ be an operator. Then the incremental operator $\tilde{\mathcal{F}}[x]$ is defined by

$$\tilde{\mathcal{F}}[x]\delta x \triangleq \mathcal{F}(x+\delta x) - \mathcal{F}x \tag{A1}$$

for all x and δx elements of $\mathfrak{X}$.

Proof of Theorem 1: First, note that

$$\mathcal{B}\mathcal{G}\hat{x} = \mathcal{B}\mathcal{G}(x+e) = \mathcal{B}\mathcal{G}x + \widetilde{(\mathcal{B}\mathcal{G})}[x]e. \tag{A2}$$

Let $\xi' \triangleq \xi + \widetilde{(\mathcal{B}\mathcal{G})}[x]e$. Then the dynamics of the closed-loop system with $u = \mathcal{G}\hat{x}$ satisfy

$$\frac{d}{dt}x = (\mathcal{Q} + (\mathcal{B}\mathcal{G}))x + \xi'; \qquad x(0) = 0 \tag{A3}$$

whereas the dynamics with $u = \mathcal{G}x$ satisfy

$$\frac{d}{dt}x = (\mathcal{Q} + \mathcal{B}\mathcal{G})x + \xi; \qquad x(0) = 0. \tag{A4}$$

Since by hypothesis (A4) is stable (finite gain stable), it is sufficient to observe that

$$\|\xi'\| \triangleq \|\xi + \widetilde{(\mathcal{B}\mathcal{G})}[x]e\| \leq \|\xi\| + \|\widetilde{(\mathcal{B}\mathcal{G})}[x]e\|$$
$$\leq \|\xi\| + \tilde{g}(\mathcal{B})\tilde{g}(\mathcal{G})\|e\| < \infty. \tag{A5}$$

Proof of Theorem 2: Let s denote the linear functional operator $s = d/dt$. From (4.4) and (4.5) it follows that

$$se = (\tilde{\mathcal{Q}}[x] - H\tilde{\mathcal{C}}[x])e - (\xi - H\theta). \tag{A6}$$

Premultiplying by P^{-1}, introducing the dummy variable z and the arbitrary constant $\epsilon > 0$, and rearranging yields

$$(P^{-1}e) = (s+\epsilon)^{-1}P^{-1}z \tag{A7a}$$
$$z = -(-\tilde{\mathcal{Q}}[x] + H\tilde{\mathcal{C}}[x] - \epsilon I)P(P^{-1}e) - (\xi - H\theta). \tag{A7b}$$

From [7, theorem 3] a sufficient condition for (A7) to be finite gain stable is the existence of an $\epsilon > 0$ such that[5]

$$(s+\epsilon)^{-1}P^{-1} \geq 0 \tag{A8a}$$
$$(-\tilde{\mathcal{Q}}[x] + H\tilde{\mathcal{C}}[x] - \epsilon I)P > 0 \tag{A8b}$$

uniformly for all $x \in \mathfrak{M}_2(R_+, R^n)$ and for all w. Parseval's theorem ensures that (A8a) holds for all $\epsilon > 0$. Define

$$\mathcal{F} \triangleq (A - \mathcal{Q} - H(C - \mathcal{C}))P + \frac{1}{2}S. \tag{A9}$$

Then in view of (5.1), a necessary and sufficient condition for (A8b) to hold is

$$\tilde{\mathcal{F}}[x] > 0 \tag{A10}$$

uniformly for all $x \in \mathfrak{M}_2(R_+, R^n)$ and for all w. Now, for all $\eta \in \mathfrak{M}_2(R_+, R^n)$

$$\begin{aligned}\tilde{\mathcal{F}}[x]\eta &\triangleq \mathcal{F}(x+\eta) - \mathcal{F}\eta \\ &= \int_x^{x+\eta} \nabla \mathcal{F}[z]\,dz \\ &= \int_0^1 \nabla \mathcal{F}[x+\rho\eta]\eta\,d\rho.\end{aligned} \tag{A11}$$[6]

So, for all $\eta \in \mathfrak{M}_2(R_+, R^n)$

[5]Actually [7, theorem 3] merely claims boundedness rather than finite gain stability. A careful review of the proofs of [7] reveals that the stronger claim of finite gain stability is justified in the present situation (cf. [9, p. 109], [20, part II]). Also, it should be noted that technically the definition of an extended normed space employed in [7] does not permit the extension of $\mathfrak{M}_2(\cdot,\cdot)$; a less restrictive definition, consistant with Zames' theory [7] and admitting $\mathfrak{M}_2$, is given in [20, part II].

[6]Integration between two points in an infinite-dimensional function space such as $\mathfrak{M}_2(R_+, R^n)$ is completely analogous to ordinary Riemann integration between two points in a finite dimensional space such as R^n—cf. [21, p. 665].

$$\langle \eta, \tilde{\mathfrak{F}}[x]\eta\rangle_\tau = \langle \eta, \int_0^1 \nabla(\tilde{\mathfrak{F}}[x])[\rho\eta]\eta\, d\rho\rangle_\tau = \int_0^1 \langle \eta, \nabla\mathfrak{F}[x+\rho\eta]\eta\rangle_\tau\, d\rho. \tag{A12}$$

Thus, a sufficient condition for (A10) and hence (A8b) to hold is $\nabla\mathfrak{F}[x]$ uniformly strongly positive; that is, uniformly for all $x \in \mathfrak{M}_2(R_+, R^n)$ and for all w

$$[(A - \nabla\mathcal{G}[x]) - H(C - \nabla\mathcal{C}[x])]P + \frac{1}{2}S > 0. \tag{A13}$$

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