Chapter 4

Sliding Mode Observers

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4.1 Introduction

Sliding mode techniques have been widely studied and developed for the control problem and observation in the occidental countries[1] since the works of Utkin [43]. As discussed by many authors [22, 40, 21, 37, 49, 50, 20, 4, 31, 24, 33], this methodology has several drawbacks in control design, adaptive control and observation. More particularly, several authors have used sliding observer for linear and nonlinear systems, and in many applications such as robotics [41, 12, 13, 28], mobile robots [5], AC motors [16, 17, 18] and converters [36].

This kind of observer is very useful and was developed for many reasons:
- to work with reduced observation error dynamics
- for the possibility of obtaining a step-by-step design
- for a finite time convergence for all the observables states
- to design, under some conditions, an observer for nonsmooth systems, and
- robustness under parameter variations is possible, if the condition (dual of the well-known matching condition) is verified.

[1]It is important to highlight the paternity and the major contribution of the Russian school in the sliding mode domain.

Here, we highlight a few advantages of the sliding observer. One advantage is the possibility to design an observer for a system with an undetermined but bounded specific variable structure, however, throughout this chapter we choose to focus our attention on widening the class of considered systems in the design of the observer.

Historically, in nonlinear control theories, the problem of a nonlinear observer design with linearization of the observation error dynamics for a class of nonlinear systems, called *the input injection form*, has been investigated ([29, 45, 46]...). Some necessary and sufficient conditions to obtain such a form are given in [46]. From this form, it is "easy" to design an observer. Unfortunately, the geometric conditions to obtain this form are very often too restrictive with respect to the system considered. Thus, in [11] we have given an extension of the results obtained in [29, 30, 35, 45, 46], for systems that can be written in an output injection form to systems which can be written in the form of the output and the output's derivative injection. We first recall this result and then we deal with a more general case, which is the *triangular observer form* [1]. Here, aiming for simplicity, we only present the case of single output system. The multi-output case may be found in [6], where the implicit triangular observer form is introduced in order to take into account the fact that the information quantity given by one output and its derivatives may change along the state space. Roughly speaking, in the nonlinear case, in the neighborhood of x_0, information about the state can be given by the output y_1 (one component of the output) and its derivative, and in another neighborhood of x_1, information can be given by y_2 (another component of the output) and its derivative. In both forms considered in this presentation, input derivatives are prohibited. Indeed, if they are allowed it is possible to use the observer form proposed in [25] and in that case a sliding observer is also widely used (see for example [34]).

As in other chapters, some recall on high order sliding mode are given [31], then for the sake of clarity we do not present the high order sliding observer [7, 3, 7]. Moreover, we deferred some technical proofs to the appendix.

We find that it is important to end this introduction with the following warning: in this chapter we omit many interesting aspects, for example, the observer design without coordinate change [14], high gain [10], and noise sensibility [47]. The subject is too large and open, to be able to squeeze it in an introductory presentation. The main purpose of this chapter is to highlight the utilities and difficulties of sliding mode technique for the observer design.

4.2 Preliminary example

In this section, the sliding observer is introduced based on a simple academic example. Let Σ be the system:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & f(x_1, x_2) \\ y & = & x_1 \end{array} \tag{4.1}$$

where $x \in \mathbb{R}^2$ and $y \in \mathbb{R}$ is the output and the function $f(x_1, x_2)$ is bounded ($|f(x_1, x_2)| < B$) but not necessary smooth, thus (4.1) is a particular case of variable structure dynamics.

One wants to observe the state x with the additional constraint to obtain the real value of x_2 in finite time. To do this, one uses a classical sliding mode observer, but completed with a new component $\tilde{x}_2$.

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + \lambda_1 sgn(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 & = & f(x_1, \tilde{x}_2) + E_1 \lambda_2 sgn(\tilde{x}_2 - \hat{x}_2) \\ \hat{y} & = & \hat{x}_1 \\ \tilde{x}_2 & = & \hat{x}_2 + E_1 \lambda_1 sgn(x_1 - \hat{x}_1) \end{array} \tag{4.2}$$

where $\hat{x}$ represents the estimated value of x and $E_1 = 1$ if $x_1 = \hat{x}_1$ else $E_1 = 0$ and sign denote the usual sign function.

From (4.1) and (4.2), the error observation ($e = x - \hat{x}$) dynamics are:

$$\begin{array}{rcl} \dot{e}_1 & = & e_2 - \lambda_1 sgn(e_1) \\ \dot{e}_2 & = & f(x_1, x_2) - f(x_1, \tilde{x}_2) - E_1 \lambda_2 sgn(\tilde{x}_2 - \hat{x}_2) \end{array} \tag{4.3}$$

Considering the nonempty manifold $S = \{e/e_1 = 0\}$ and the Lyapunov function $V = \frac{1}{2}e_1^2$, one proves the attractivity of S as follows. One gets: $\dot{V} = e_1 e_2 - \lambda_1 e_1 sgn(e_1)$, which verifies the inequality $\dot{V} < 0$ when λ_1 is chosen such that $\lambda_1 > |e_2|_{max}$ (where $|e|_{max}$ denotes the maximal value of $e, \ \forall\, t \in [0, \infty]$). As one uses a *sgn* function and as the Lyapunov function V is decreasing, one obtains the convergence to the sliding surface $S = 0$ in finite time t_0 (and moreover, we have $|e|_{max} = |e|_{max}^{t_0}$ and $|e|_{max}^{t_0}$ is the maximal value of $e, \ \forall\, t \in [0, t_0]$). Thus, for $\lambda_1 > |e_2|_{max}$, $\hat{x}_1$ converges to x_1 in finite time and remains equal to x_1 for $t > t_0$.

Moreover, one also has that $\dot{e}_1 = 0 \ \forall\, t > t_0$, so that from (4.3),

$$e_2 = \lambda_1 sgn(e_1) \tag{4.4}$$

Therefore, the observer output, $\tilde{x}_2 = \hat{x}_2 + \lambda_1 sgn(e_1)$ is equal to $x_2 \ \forall\, t > t_0$.

Remark 31 *This is obviously only true without any noise measurement, but this difficulty may be partially overcome by a sgn function modification (see [47] for analysis and design of observer with respect to noise) or by high order sliding mode [31].*

Up to now, we proved for the system (4.1) that the observer (4.2) is suitable to give all the values of the state in finite time.

The condition $\lambda_1 > |e_2|_{max}$ can only be verified if e_2 has stable dynamics, which is fulfilled after t_0 for $\lambda_2 > 0$, where we have

$$\dot{e}_2 = f(x_1, x_2) - f(x_1, \tilde{x}_2) - E_1\lambda_2 sgn(\tilde{x}_2 - \hat{x}_2)$$

with $\tilde{x}_2 = x_2$ and $E_1 = 1$ then

$$\dot{e}_2 = -\lambda_2 sgn(e_2)$$

Therefore, one gets $|e_2|_{max}^{t_0}$, which is bounded by the way that t_0 and $f(x_1, x_2)$ are bounded. The observer (4.2) with assumptions $\lambda_1 > |e_2|_{max}^{t_0}$ and $\lambda_2 > 0$ ensures a finite time convergence of (e_1, e_2) to $(0, 0)$.

Remark 32 *The time t_0 can be very short because it is natural to initialize $\hat{x}_1 = x_1$.*

4.3 Output and output derivative injection form

Following, we recall some classical results on nonlinear observer theory.

4.3.1 Nonlinear observer

First of all, we recall the definition of *observability indices*.

Definition 33 *[29] Let the system*

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{4.5}$$

which is observable at x_0 if there exists a neighborhood $\mathcal{U}$ of x_0 and p-tuple of integers $(\mu_1, ..., \mu_p)$ such that

1) $\mu_1 \geq \mu_2 \geq ... \geq \mu_p \geq 0$ and $\sum_{i=1}^{p} \mu_i = n$.

2) After suitable reordering of the h_i at each $x \in \mathcal{U}$, the n row vectors $\left\{L_f^{j-1}(dh_i) : i = 1, ..., p;\ j = 1, ..., \mu_i\right\}$ are linearly independent.

3) If $l_1, ..., l_p$ satisfies (i) and after suitable reordering the n row vectors $\{L_f^{j-1}(dh_i) : i = 1, ..., p;\ j = 1, ..., l_i\}$ are linearly independent at some $x \in \mathcal{U}$

then $(l_1, ..., l_p) \geq (\mu_1, ..., \mu_p)$ in the lexicographic ordering $[(l_1 > \mu_1)$ or $(l_1 = \mu_1$ and $l_2 > \mu_2)$ or... or $(l_1 = \mu_1, ..., l_p = \mu_p)]$. The integers $(\mu_1, ..., \mu_p)$ are called the observability indices at x_0.

Remark 34 *In the nonlinear case, the previous notion of observability index is local. In the linear case, this notion is global.*

As it is shown in [29, 30, 45], an interesting nonlinear systems is the *output injection form* without forced terms:

$$\begin{aligned} \dot{x} &= Ax + \phi(y) \\ y &= Cx \end{aligned} \tag{4.6}$$

where:

$$A = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_p \end{array}\right)$$

$$A_i \text{ is a } \in \mathbb{R}^{\mu_i \times \mu_i} \text{ matrix } = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{array}\right) \tag{4.7}$$

$$\text{and} \quad \left(\begin{array}{c|c|c} C_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & C_p \end{array}\right)$$

C_i is a line vector $\in \mathbb{R}^{\mu_i}$, such that : $C_i = (1, 0, ..., 0)$.

This is interesting because for such a class, one can design an observer that allows us to obtain an observation error with stable linear dynamics.

In fact, for the nonlinear observable system:

$$\begin{aligned} \dot{\xi} &= f(\xi) \\ y &= h(\xi) \end{aligned} \tag{4.8}$$

where f and h are smooth functions, necessary and sufficient conditions for the existence of a diffeomorphism $x = \Phi(\xi)$ to transform the system (4.8) into (4.6) are given in [46].

Theorem 35 *[46] There exists a change of coordinates transforming (4.8) into (4.6) only if there exists a $p-$tuple of integers $(\mu_1, ..., \mu_p)$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_p$ such that we have the following:*

1) If one denotes (with a possible reordering of the h_i)

$$Q = \left\{ L_f^{j-1}(dh_i) : \ i = 1, ..., p; \ j = 1, ..., \mu_i \right\}$$

then dim span $Q = n$ in a neighborhood of ξ^0.

2) If one denotes for $j = 1, ..., p$,

$$Q_j = \left\{ L_f^{k-1}(dh_i) : \begin{array}{l} i = 1, ..., p; \\ k = 1, ..., \mu_j \end{array} \right\} - \left\{ L_f^{\mu_j - 1}(dh_j) \right\}$$

then span Q_j = span $Q \cap Q_j$ for $j = 1, ..., p$.

Theorem 36 *[46] There exists a change of coordinates transforming (4.8) to (4.6) if and only if 1. and 2. in the previous Theorem hold and, moreover, if there exists vector fields $g^1, ..., g^p$ satisfying:*

$$L_{g^i} L_f^{l-1}(h_j) = \delta_{i,j}\delta_{l,\mu_i}, \ \ i, j = 1, ..., p, \ l = 1, ..., \mu_i$$

such that: $[ad^k_{(-f)} g^i, ad^l_{(-f)} g^j] = \mathbf{0}$for $i, j = 1, ..., p$; $k = 0, ..., \mu_i - 1$; $l = 0, ..., \mu_j - 1$.

Thus, it immediately follows:

Corollary 37 *The conditions of Theorem 36 are sufficient to construct an observer that is asymptotically locally stable.*

4.3.2 Sliding observer for output and output derivative nonlinear injection form

In this section, one first constructs an asymptotically stable observer for the following class of systems called *output and output derivative nonlinear injection form*:

$$\begin{array}{rcl} \dot{x}_i & = & A_i x + \phi_i(y, \dot{y}) \\ y_i & = & x_{i,1} \\ \dot{y}_i & = & x_{i,2} \end{array} \qquad \text{for } i = 1, ..., p \tag{4.9}$$

with

$$\phi_i(y,\dot{y}) = \begin{pmatrix} \phi_{i,1}(y) \\ \phi_{i,2}(y,\dot{y}) \\ \vdots \\ \phi_{i,\mu_p}(y,\dot{y}) \end{pmatrix}$$

and all A_i matrix are of appropriated dimensions. Secondly, one exhibits the necessary and sufficient conditions under which the system (4.8) may be rewritten as (4.9). For the sake of simplicity, one introduces the following notations:

$$\begin{array}{rcl} x_i & = & (x_{i,1}, x_{i,2}, ..., x_{i,\mu_i})^T \\ \tilde{x} & = & (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_p)^T \\ \hat{x}_i & = & (\hat{x}_{i,1}, \hat{x}_{i,2}, ..., \hat{x}_{i,\mu_i})^T \qquad \text{for} \quad i = 1,..,p \end{array}$$

where $\tilde{x}_i = \hat{x}_{i,2} + E_1\lambda_{i,1}sgn(y_i - \hat{x}_{i,1})$ and $E_1 = 1$ if $(x_{1,1} - \hat{x}_{1,1})=$ $...=(x_{1,p} - \hat{x}_{1,p})=0$, else $E_1=0$.

Let us construct for the system (4.9) the sliding observer:

$$\begin{array}{rcl} \dot{\hat{x}}_{i,1} & = & \hat{x}_{i,2} + \phi_1(y) + \lambda_{i,1}sgn(y_i - \hat{y}_i) \\ \dot{\hat{x}}_{i,2} & = & \hat{x}_{i,3} + \phi_2(y,\dot{\tilde{y}}) + E_1\lambda_{i,2}sgn(y_i - \hat{y}_i) \\ \vdots & = & \vdots \\ \dot{\hat{x}}_{i,\mu_i-1} & = & \hat{x}_{i,\mu_i} + \phi_{\mu_i-1}(y,\dot{\tilde{y}}) + E_1\lambda_{i,\mu_i-1}sgn(y_i - \hat{y}_i) \\ \dot{\hat{x}}_{i,\mu_i} & = & \phi_{\mu_i}(y,\dot{\tilde{y}}) + E_1\lambda_{i,\mu_i}sgn(y_i - \hat{y}_i) \\ \hat{y}_i & = & \hat{x}_{i,1} \end{array} \tag{4.10}$$

for $i = 1,...,p$ where: $\dot{\tilde{y}}_i \triangleq \hat{x}_{i,2} + E_1\lambda_{i,1}sgn(y_i - \hat{y}_i)$
From this, one deduces a part of the error's observation dynamic ($e_{i,1} = (y_i - \hat{x}_{i,1})$ and $e_{i,2} = \dot{y}_i - \hat{x}_{i,2}$):

$$\dot{e}_{i,1} = e_{i,2} + \lambda_{i,1}sgn(e_{i,1})$$

Therefore, using the same method as in the previous section one obtains:

Theorem 38 *Under the conditions:*

1) $\lambda_{i,1} > |e_{2,i}|_{max}$ *for* $i = 1,...,p$.

2) *All the* $\lambda_{i,j}$ $i = 1,...,p,$ $j = 2,...,\mu_i$ *are such that* $\left[sI - (A_i + \frac{\Lambda_i}{\lambda_{i,1}}u_1)\right]$ *is a Hurwitz polynomial. Where* $u_1 = (1,0,...,0)^T$ *and* A_i *is the*

$(\mu_i - 1) \times (\mu_i - 1)$ matrix defined by

$$A_i \text{ is a } \in \mathbb{R}^{(\mu_i-1)\times(\mu_i-1)} \text{ matrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The observer (4.10) gives, in finite time t_0, the convergence of $\hat{y}$ (respectively $\dot{y}$) to y (respectively to $\dot{\tilde{y}}$), and an asymptotic linear stable observation error dynamics on the sliding surface ($e_{i,1} = 0$).

Proof The dynamics of the observation error are

$$\dot{e}_i = A_i e_i + \phi(y, \dot{y}) - \phi(y, \dot{\tilde{y}}) - \Lambda_i sgn(y_i - \hat{y}_i)$$

for $i = 1, ..., p$. It is clear that, after a finite time t_0, one has $\dot{y} = \dot{\tilde{y}}$, so $\phi(y, \dot{y}) - \phi(y, \dot{\tilde{y}}) = 0$. So that, for $\forall\, t > t_0$ the error dynamics will be on the reduced manifold $(e_{i,1} = 0)$, $\forall\, i \in \{1, ..., p\}$, and given by

$$\dot{\bar{e}}_i \quad = \quad A_i \bar{e}_i - \Lambda_i \frac{e_{i,2}}{\lambda_{i,1}} \qquad \text{for } i = 1, ..., p \tag{4.11}$$

with $\bar{e}_i = (e_{i,2}, e_{i,3}, ..., e_{i,\mu_i})$ which is linear. If $\left[sI - (A_i + \frac{\Lambda_i}{\lambda_{i,1}} u_1)\right]$ is Hurwitz, this dynamic is asymptotically stable.

One has shown that using a sliding mode observer (4.10), the system (4.9) may be, under an appropriate choice of $\lambda_{i,j}$, observed with a linear asymptotic stable observation error dynamics (4.11).

In the next proposition, one characterizes the observability indices of the output $\bar{y} \stackrel{\triangle}{=} (y, \dot{y}) = (h, L_f h)$.

Proposition 39 *Considering the system (4.8) with the extended output: $\bar{y} = (y, \dot{y}) = (h, L_f h)$:*

$$\begin{aligned} \dot{\xi} &= f(\xi) \\ \bar{y} &= (h(\xi), L_f h(\xi)) \end{aligned} \tag{4.12}$$

the indices of observability become:

$$\bar{\mu}_i = \begin{cases} 1 & if\ i \in \{1, .., p\} \\ & one\ has\ \bar{y}_i = y_j\ with\ j = i \\ \mu_j - 1 & if\ i \in \{p+1, .., 2p\} \\ & one\ has\ \bar{y}_i = \dot{y}_j\ with\ j = i - p \end{cases}$$

where μ_i is the observability indices of the output y_i in the system (4.8).

For the proof see the appendix, page 123.

Remark 40 *The necessary and sufficient conditions to obtain output and output derivative form are the same as those in Theorem 35 for the extended output* $\bar{y} = (y, \dot{y})$.

From the last remark, necessary and sufficient conditions for the existence of a diffeomorphism transforming (4.8) into (4.12) are given by applying Theorem 36 to system (4.12) rewritten only in terms of the real output y.

Theorem 41 *There exists a change of coordinates transforming (4.12) into (4.9) if and only if*

1) *If one denotes (with a possible reordering of the* h_i *)*

$$Q = \left\{ L_f^{j-1}(dh_i) \text{ with } i = 1, ..., p \text{ and } j = 1, ..., \mu_i \right\}$$

then dim span $Q = n$ *in a neighborhood of* ξ^0.

2) *If one denotes for* $j = 1, ..., p$

$$Q_j = \left\{ L_f^{k-1}(dh_i) \quad \begin{matrix} i = 1, ..., p \\ k = 1, ..., \mu_j \end{matrix} \right\} - \left\{ L_f^{\mu_j - 1}(dh_j) \right\}$$

then for $j = 1, ..., p$ *span* $Q_j =$ *span* $Q \cap Q_j$

3) *There exists vector fields* $\bar{g}^1, \bar{g}^2, ..., \bar{g}^{2p}$ *satisfying:*

$$L_{\bar{g}^i} L_f^{l-1}(h_j) = \delta_{i,j}\delta_{l,\mu_i}, \quad \text{with} \left\{ \begin{array}{l} i, j = 1, ..., p, \\ l = 1, ..., \mu_i \end{array} \right.$$

$$L_{\bar{g}^i}(h_j) = \delta_{i,j+p}, \quad \text{with} \left\{ \begin{array}{l} i = p+1, ..., 2p, \\ j = 1, ..., p \end{array} \right.$$

$$\text{and } L_{\bar{g}^i}(L_f(h_j)) = 0, \quad \text{with} \left\{ \begin{array}{l} i = p+1, ..., 2p, \\ j = 1, ..., p \end{array} \right.$$

4) *Setting :*

$$\bar{\Delta} = \left\{ ad^k_{(-f)}\bar{g}^i, \quad \begin{matrix} i = 1, .., p \\ k = 0, .., \mu_i - 2 \end{matrix} \right\} \cup \{\bar{g}^i, \ i = p+1, .., 2p\}$$

$$\forall u, v \in \bar{\Delta}, \qquad u \neq v \Rightarrow [u, v] = 0$$

For the proof see the appendix, page 124.

Remark 42 *From the proof of Theorem 41, one can see that the Definition of* $\bar{g}^i$ *for* $i = 1, ..., p$ *is the same as the definition of* g^i. *However, condition 3. is less restrictive than the one given in Theorem 35.*

Example 43 *Let us consider the following system:*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + x_2^2 \\ \dot{x}_3 &= -x_3 - x_2 - x_1 \\ y &= x_1\end{aligned} \tag{4.13}$$

which is in output and output derivative nonlinear injection form and can not be transformed into output injection form. In fact, as defined in Theorem 35, the vector g^1 is such that:

$$\begin{aligned}L_{g^1}(x_1) = L_{g^1}(x_2) = 0 \\ L_{g^1}(x_3 + x_2^2) = 1\end{aligned} \tag{4.14}$$

So, $g^1 = (0,0,1)^T$, $ad^1_{(-f)}g^1 = (0,1,-1)^T$ and $ad^2_{(-f)}g^1 = (1, 2x_2 - 1, 0)^T$.

The Lie brackets of these vectors are equal to

$$\begin{aligned}\left[g^1, ad^1_{(-f)}g^1\right] = \left[g^1, ad^2_{(-f)}g^1\right] = 0 \\ \left[ad^1_{(-f)}g^1, ad^2_{(-f)}g^1\right] = (0,2,0)^T \neq 0\end{aligned}$$

Consequently, this system does not verify the conditions of Theorem 35. Looking now at the conditions of Theorem 38, one has for the vectors $\bar{g}^1$ and $\bar{g}^2$:

$$\begin{aligned}L_{\bar{g}^1}(x_1) = L_{\bar{g}^1}(x_2) = 0,\ L_{\bar{g}^1}(x_3 + x_2^2) = 1 \\ L_{\bar{g}^2}(x_1) = 1,\ \ L_{\bar{g}^2}(x_2) = 0\end{aligned}$$

So, $\bar{g}^1 = (0,0,1)^T$, $ad^1_{(-f)}\bar{g}^1 = (0,1,-1)^T$, and $\bar{g}^2 = (1,0,*)^T$. Then if one chooses $* = 0$ for example, one obtains:

$$\left[\bar{g}^1, ad^1_{(-f)}\bar{g}^1\right] = \left[\bar{g}^2, \bar{g}^1\right] = \left[\bar{g}^2, ad^1_{(-f)}\bar{g}^1\right] = 0$$

Thus, this system verifies all the conditions of Theorem 38. Choosing $z_1 = x_1$; $z_2 = x_2$; $z_3 = x_2 + x_3$, one obtains in the new coordinates the following system:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \phi_2(z_1, z_2) \\ \dot{z}_3 &= \phi_3(z_1, z_2) \\ y &= z_1\end{aligned}$$

Remark 44 *Every system in the form of (4.6) is obviously on the form (4.9). One important consequence of the previous remark and the example is that the conditions of Theorem 35 imply conditions of Theorem 38, but the converse is false.*

In the next section we consider an actuated system but for the sake of simplicity only in a single input single output (SISO) form.

4.4 Triangular input observer form

Let us consider the following SISO analytic system Σ

$$\begin{aligned} \dot{x} &= f(x) + g(x,u) \\ y &= h(x) \end{aligned} \tag{4.15}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output and f, g, h are analytical function vectors of appropriate dimensions. Moreover for any $x \in \Re^n$ the function $g(x,0)$ is equal to zero and the system (4.15) is assumed bounded input bounded state in finite time. In order to transform (4.15) in a triangular input observer form, we modified the classical *observation rank condition*:

Condition 45

$$rank \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \\ dL_f^{\infty} h \end{pmatrix} = n$$

where L_f denotes the classical Lie derivative in f and dh is the classical one form.

Remark 46 *Condition 45 is the classical one for an autonomous system. In the nonlinear context, we can't refer to the Cayley-Hamilton theorem.*

But in the next we assume

Condition 47

$$rank \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{pmatrix} = n$$

From condition 47 it is known that the codistribution

$$\Omega^i = span\{dh, ..., dL_f^i h\} \qquad 0 \leq i \leq n-1$$

is involutive. We also need the following condition

Condition 48 *The vector field g verifies for any $u \in \mathbb{R}$*

$$dL_g L_f^i h \in \Omega^i \qquad \forall i \in \{0, ..., n-1\}$$

Now we can set the following Theorem :

Theorem 49 *System (4.15) may be transformed, by diffeomorphism, in the neighborhood of x in a triangular input observer form*

$$\begin{aligned}
\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{pmatrix} &= \begin{pmatrix} \xi_2 + \bar{g}_1(\xi_1, u) \\ \xi_3 + \bar{g}_2(\xi_1, \xi_2, u) \\ \vdots \\ \xi_n + \bar{g}_{n-1}(\xi_1, ..., \xi_{n-1}, u) \\ \bar{f}_n(\xi) + \bar{g}_n(\xi, u) \end{pmatrix} \\
y &= \xi_1
\end{aligned} \tag{4.16}$$

with $\bar{g}_i(., u = 0) = 0$ for any $i \in \{1, ..., n\}$, if and only if conditions 47 and 48 hold in the neighborhood of x.

For the proof see the appendix, page 125.

4.4.1 Sliding mode observer design for triangular input observer form

From the work of Drakunov and Utkin [14, 15] and our previous work [28, 16, 6], we propose the *sliding observer for triangular input observer form*

$$\begin{pmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \\ \vdots \\ \dot{\hat{\xi}}_{n-1} \\ \dot{\hat{\xi}}_n \end{pmatrix} = \begin{pmatrix} \hat{\xi}_2 + \bar{g}_1(\xi_1, u) + \lambda_1 sgn(\xi_1 - \hat{\xi}_1) \\ \hat{\xi}_3 + \bar{g}_2(\xi_1, \tilde{\xi}_2, u) + \lambda_2 sgn_1(\tilde{\xi}_2 - \hat{\xi}_2) \\ \vdots \\ \hat{\xi}_n + \bar{g}_{n-1}(\xi_1, \tilde{\xi}_2 ..., \tilde{\xi}_{n-1}, u) + \lambda_{n-1} sgn_{n-2}(\tilde{\xi}_{n-1} - \hat{\xi}_{n-1}) \\ \bar{f}_n(\xi_1, \tilde{\xi}_2 ..., \tilde{\xi}_n) + \bar{g}_n(\xi_1, \tilde{\xi}_2 ..., \tilde{\xi}_n, u) + \lambda_n sgn_{n-1}(\tilde{\xi}_{n-1} - \hat{\xi}_n) \end{pmatrix} \tag{4.17}$$

where

$$
\begin{aligned}
\tilde{\xi}_2 &= \hat{\xi}_2 + \lambda_1 sgn_1(\xi_1 - \hat{\xi}_1) \\
\tilde{\xi}_3 &= \hat{\xi}_3 + \lambda_2 sgn_2(\tilde{\xi}_2 - \hat{\xi}_2) \\
\vdots &= \\
\tilde{\xi}_n &= \hat{\xi}_n + \lambda_n sgn_{n-1}(\tilde{\xi}_{n-1} - \hat{\xi}_{n-1})
\end{aligned}
$$

and the $sgn_i(\xi)$ function denotes the usual sgn function but with a low pass filter of the ξ variable [15] and an anti-peaking structure [6]. This anti-peaking structure follows the idea that we do not inject the observation error information before reaching the sliding manifold linked with this information (i.e., $sign_i = E_i sign$, with $E_i = 1$ if $E_1 = ... = E_{i-1} = 1$ and $\xi_1 - \hat{\xi}_1 = 0$ else $E_i = 0$). Moreover we reach the manifold one by one. Doing this we obtain a "high gain" dynamic (i.e., see the equivalence between the sliding mode and the high gain [32]) of dimension one and consequently we do not have a peaking phenomena [42]. More precisely $sgn_i(.)$ is equal to zero if their exists $0 < j < i-1$ such that $\tilde{\xi}_j - \hat{\xi}_j \neq 0$ (by definition $\tilde{\xi}_1 = \xi_1$), else $sgn_i(.)$ is equal to the usual $sgn(.)$ function. In the observer structure, this particular sgn function allows that $\tilde{\xi}_i - \hat{\xi}_i$ converges to zero if all the $\tilde{\xi}_j - \hat{\xi}_j$ with $j < i$ have converged to zero before.

Theorem 50 *Considering a bounded input bounded state (BIBS) in finite time system (4.16) and observer (4.17), for any initial state $\xi(0)$, $\hat{\xi}(0)$ and any bounded input u, there exists a choice of λ_i such that the observer state $\hat{\xi}$ converges in finite time to ξ.*

Proof From (4.16) and (4.17) and considering the initial state condition such that $\xi_1(0) \neq \hat{\xi}_1(0)$ (if this is not the case, we directly move on to the next step of the proof). Thus we are in the

• **first step** of our proof and we obtain the following observation error dynamics $e = \xi - \hat{\xi}$.

$$
\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 sgn(\xi_1 - \hat{\xi}_1) \\ e_3 + \bar{g}_2(\xi_1, \xi_2, u) - \bar{g}_2(\xi_1, \hat{\xi}_2, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, ..., \xi_{n-1}, u) - \bar{g}_{n-1}(\xi_1, \hat{\xi}_2, ..., \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, \hat{\xi}_2, ..., \hat{\xi}_n) + \bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, \hat{\xi}_2, ..., \hat{\xi}_n, u) \end{pmatrix}
$$

Thus as the input u is bounded the state ξ does not go to infinity in finite time. Moreover if $\hat{\xi}_1$ is bounded all the states of the observer are also bounded during step 1. Consequently the observation error state is also bounded. Now, setting $V_1 = \frac{e_1^2}{2}$, we have

$$\dot{V}_1 = e_1(e_2 - \lambda_1 sgn(e_1))$$

Thus choosing $\lambda_1 >| e_2 |_{max}$ the observation error e_1 goes to zero in finite time t_1. Moreover, if after t_1 the observation error stays equal to zero (i.e., $\lambda_1 >| e_2 |_{max}$) we have $e_2 = \lambda_1 sgn(\xi_1 - \hat{\xi}_1)$ and consequently $\tilde{\xi}_2 = \xi_2$. Now we pass to the:

• **second step**. Here, we ensure that the observation error e_2 is bounded in order to remain on the manifold $e_1 = 0$. Moreover, we want to reach the submanifold $e_1 = e_2 = 0$. Using the same argument as in [14, 15] the equivalent vector is obtained in finite time via a low past filtering of $\lambda_1 sgn(\xi_1 - \hat{\xi}_1)$ which is equal to e_2. Thus, as at t_1, we have $e_1 = 0$, and the observation error is now equal to

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 sgn(\xi_1 - \hat{\xi}_1) = 0 \\ e_3 - \lambda_2 sgn(\xi_2 - \hat{\xi}_2) \\ e_4 + \bar{g}_3(\xi_1, \xi_2, \xi_3, u) - \bar{g}_3(\xi_1, \xi_2, \hat{\xi}_3, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, ..., \xi_{n-1}, u) - \bar{g}_{n-1}(\xi_1, \xi_2, \hat{\xi}_3, ..., \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, \xi_2, ..., \hat{\xi}_n) + \bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, \xi_2, ..., \hat{\xi}_n, u) \end{pmatrix}$$

Setting $V_2 = \frac{e_1^2}{2} + \frac{e_2^2}{2}$, we obtain

$$\dot{V}_2 = e_1(e_2 - \lambda_1 sgn(e_1)) + e_2(e_3 - \lambda_2 sgn(e_2))$$

Moreover, if the condition $\lambda_1 >| e_2 |_{max}$ holds for $t > t_1$, we have $e_1 = 0$ and $e_2 - \lambda_1 sgn(e_1) = 0$, thus we find

$$\dot{V}_2 = e_2(e_3 - \lambda_2 sgn(e_2))$$

Consequently e_2 goes to zero in finite time $t_2 > t_1$ if $\lambda_2 >| e_3 |_{max}$. Moreover, from $\dot{V}_2$ we obtain that the observation error is strictly decreasing during the period of time $[t_1, t_2]$. This implies that the condition on λ_1 is verified after t_1 if it is verified before t_1. Moreover as the input is bounded, the state ξ stays bounded during the period $[0, t_2]$ and from the structure of the observation error the dynamics e is also bounded and consequently $\hat{\xi}$ is too.

Now let us assume that we are at the step $j < n$. This step starts at time t_{j-1} and at t_{j-1}, all the $e_k = 0$ and all the conditions on λ_k are

verified for $k < j$. Thus, we proceed to
• **step** j. The observation error dynamic is equal to

$$\begin{pmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{j-1} \\ \dot{e}_j \\ \dot{e}_{j+1} \\ \vdots \\ \dot{e}_{n-1} \\ \\ \dot{e}_n \\ \\ \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 sgn(\xi_1 - \hat{\xi}_1) = 0 \\ \vdots \\ e_j - \lambda_{j-1} sgn(\xi_{j-1} - \hat{\xi}_{j-1}) = 0 \\ e_{j+1} + \lambda_j sgn(\xi_j - \hat{\xi}_j) \\ e_{j+2} + \bar{g}_{j+1}(\xi_1, ..., \xi_{j+1}, u) - \bar{g}_{j+1}(\xi_1, ..., \xi_j, \hat{\xi}_{j+1}, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, ..., \xi_{n-1}, u) \\ -\bar{g}_{n-1}(\xi_1, ..., \xi_j, \hat{\xi}_{j+1}, ..., \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, ..., \xi_j, \hat{\xi}_{j+1}, ..., \hat{\xi}_n) \\ +\bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, ..., \xi_j, \hat{\xi}_{j+1} ..., \hat{\xi}_n, u) \end{pmatrix}$$

Setting $V_j = \sum_{i=1}^{j} \frac{e_i^2}{2}$ we deduce from $e_k = 0 \; \forall i < j$ that

$$\dot{V}_j = e_j(e_{j+1} - \lambda_j sgn(e_j))$$

Consequently e_j goes to zero in finite time $t_j > t_{j-1}$ if $\lambda_j > \mid e_{j+1} \mid_{max}$ and all λ_k conditions are verified for $k < j$. As the input is bounded ξ is bounded and from the observer structure $\hat{\xi}_j$ is also bounded during the period $[0, t_j]$. It follows that e_j is bounded and we can find λ_j such that $\lambda_j > \mid e_{j+1} \mid_{max}$ is verified. Moreover, as e_j is decreasing during the period $[t_{j-1}, t_j]$, $\lambda_{j-1} > \mid e_j \mid_{max}$ is verified during this period and therefore all the e_k remain equal to zero for any $k < j$.
Now we go to :
• **step** n. This step starts at the time t_{n-1} and at this time $e_k = 0$ for any $k < n$. Thus we obtain the following observation error dynamics

$$\begin{pmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 + \lambda_1 sgn(\xi_1 - \hat{\xi}_1) = 0 \\ \vdots \\ e_n + \lambda_{n-1} sgn(\xi_{n-1} - \hat{\xi}_{n-1}) = 0 \\ \lambda_n sgn(\xi_n - \hat{\xi}_n) \end{pmatrix}$$

Setting $V_n = \sum_{i=1}^{n} \frac{e_i^2}{2}$ we deduce from $e_k = 0 \; \forall i < n$ that

$$\dot{V}_n = e_n \left[-\lambda_n sgn(e_n) \right]$$

So, e_n go to zero in finite time $t_n > t_{n-1}$ for any $\lambda_n > 0$ and if all the conditions on the λ_k for $k < n$ are verified after t_{n-1}. Condition on λ_{n-1} is always verified because e_n is decreasing after t_{n-1} and by induction all conditions follow.

4.4.2 Observer matching condition

It is well known from the work [19] that roughly speaking, a condition in order to reject a perturbation, is that the perturbation act in the same direction of the control.

In the same manner of thinking, for observer design we obtain the condition in order to observe the state under unknown perturbation. Consider the linear observable bounded perturbed system:

$$\dot{x} = Ax + Bu + Pw \tag{4.18}$$

and the output equation is $y = Cx$ with $y \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in [-B_w,\ B_w]$

$$\mathcal{O}(A, C) = \begin{pmatrix} C \\ \vdots \\ CA^{n-2} \end{pmatrix} = n$$

A condition in order to cancel the perturbation effect on the state observation is that

$$\begin{pmatrix} C \\ \vdots \\ CA^{n-2} \end{pmatrix} P = 0$$

which is called the *observer matching condition.*

Remark 51 *Necessity is obvious such that the perturbation derivative time does not act on the state observations.*

Sufficiency is clear: considering for example, an observer for triangular input observer.

Generalizing the previous observer matching condition to the bounded input bounded state single input single output (BIBS–SISO) local weakly observable nonlinear perturbed system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w := F(x, u) + p(x)w \\ y &= h(x) \end{aligned} \tag{4.19}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and the bounded perturbation $w \in [-B_w, B_w]$, and f, g, p are functions vector fields of appropriate dimensions.

We immediately obtain the following sufficient conditions in order to reject the perturbation effect on the observer.

Proposition 52 *If the system (4.19) without perturbation verifies conditions (47) and Condition 48 of Theorem 49 and the observer matching condition*

$$\begin{pmatrix} dh \\ dL_F h \\ \vdots \\ dL_F^{n-2} h \end{pmatrix} p(x) = 0 \tag{4.20}$$

in the neighborhood of x, and where the Lie derivative is done with respect to x and u. Then it is possible to locally design an observer which estimates all state components and does this in both cases: with and without perturbation.

Proof The proof is a direct consequence of Theorem 49 and sliding mode triangular observer design.

4.5 Simulations and comments

Let us consider the following system Σ which is in the triangular input observer form

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 u \\ \dot{x}_2 &= x_3 + x_2 x_1 u \\ \dot{x}_3 &= -3x_3 - 3x_2 - x_1 - x_3^3 - u \\ y &= x_1 \end{aligned} \tag{4.21}$$

For this system, the observer 4.17 takes the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - x_1^3 u + \lambda_1 sgn(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + \tilde{x}_2 x_1 u + \lambda_2 sgn_1(\tilde{x}_2 - \hat{x}_2) \\ \dot{\hat{x}}_3 &= -3\tilde{x}_3 - 3\tilde{x}_2 - x_1 - \tilde{x}_3^3 - u + \lambda_3 sgn_2(\tilde{x}_3 - \hat{x}_3) \end{aligned} \tag{4.22}$$

$$y = x_1 \tag{4.23}$$

with $\tilde{x}_2 = \hat{x}_2 + \lambda_1 sgn_1(x_1 - \hat{x}_1)$ and $\tilde{x}_3 = \hat{x}_3 + \lambda_2 sgn_2(\tilde{x}_2 - \hat{x}_2)$, and where sgn_i functions are designed as noted in Section 4.3.

This approach has been tested by simulation with the following initial conditions $x = (1, 0.5, 0.5)^T$ and $\hat{x} = (0, 0, 0)^T$. Moreover, we have chosen a first-order low pass filter with a cut frequency equal to $100Hz$ and observation gain λ_1, λ_2, and λ_3 respectively equal to 4, 2, and 2. Moreover the function "*sgn*" is approximated by a saturation function with a slow rate equal to 10^4.

In Figure 4.1, we see that $\hat{x}_1$ reaches x_1 in finite time $\simeq 0.25s$. In Figure 4.2, we see that $\hat{x}_2$ also reaches x_2 in finite time $\simeq 0.75s$. But $\hat{x}_2$ will only reach x_2 when $\hat{x}_1$ will be equal to x_1. In Figure 4.3, we see that $\hat{x}_3$ reaches x_3 in finite time $\simeq 1s$.

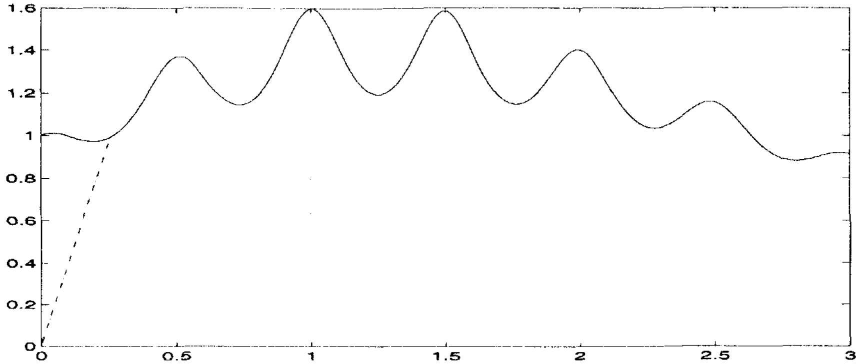

Figure 4.1: x_1(-) and $\hat{x}_1$ (-.)

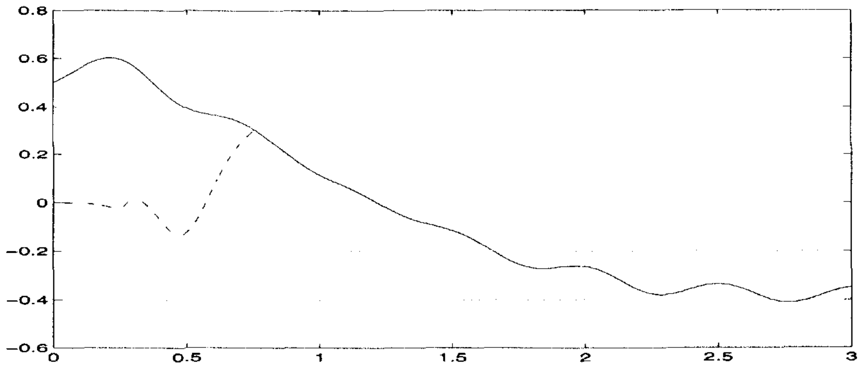

Figure 4.2: x_2(-) and $\hat{x}_2$ (-.)

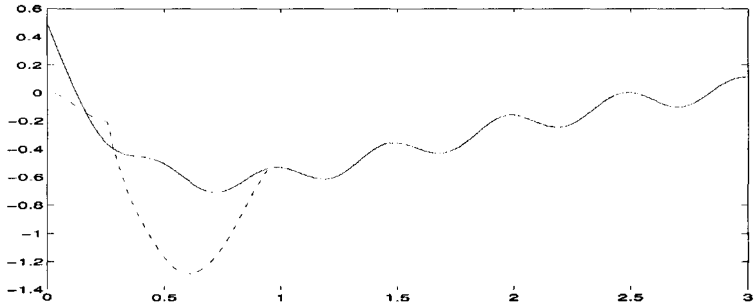

Figure 4.3: x_3(-) and $\hat{x}_3$ (-.)

Now, starting from the same initial conditions, we add an output noise in order to show the behavior of the observer in this case. In [6], following the work of Yaz and Azemi [47], the author proposed to use a saturation function with dead zone for observer in the case of the extended injection form. This reduces the observer sensitivity to the noise, but we were obliged to change the observer gain as follows $\lambda_1 = \lambda_2 = \lambda_3 = 4$ in order to recover a time response quite similar to the previous simulation.

In Figures 4.4, 4.5, 4.6 and 4.7, we see that the observer state $\hat{x}$ reaches the neighborhood of the system state x in finite time. But we also see that the noise is not totally suppressed in the observer. We can reduce this noise with some minor modifications by introducing an asymptotic gain or a *sgn* function modified with respect to the noise output knowledge [47], for example.

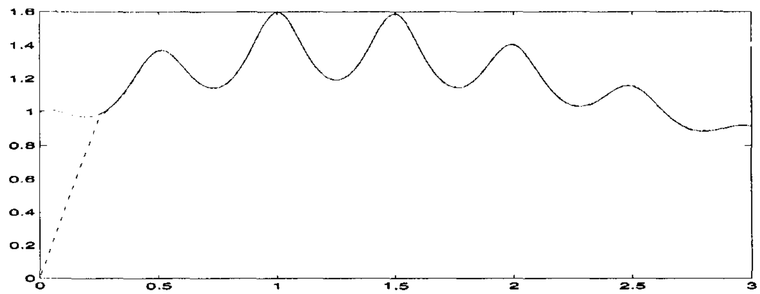

Figure 4.4: x_1(-) and $\hat{x}_1$ (-.)

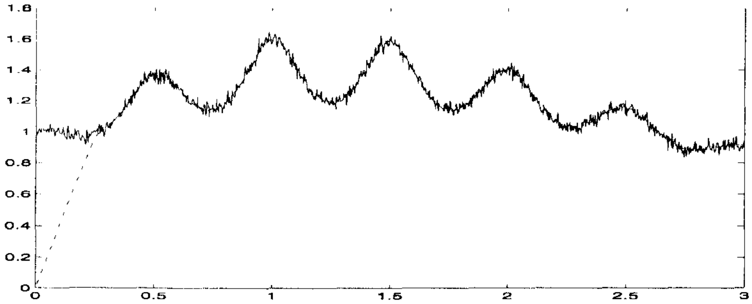

Figure 4.5: Measured x_1(-) and $\hat{x}_1$ (-.)

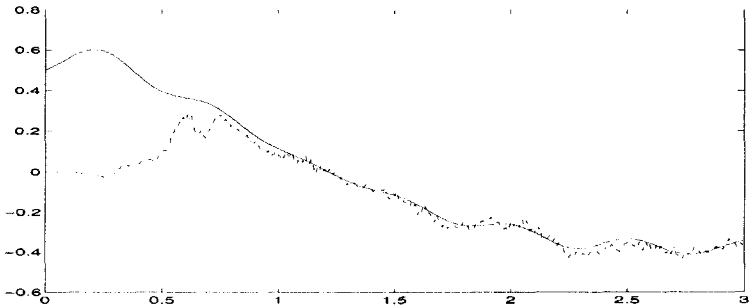

Figure 4.6: x_2 (-) and $\hat{x}_2$ (-.)

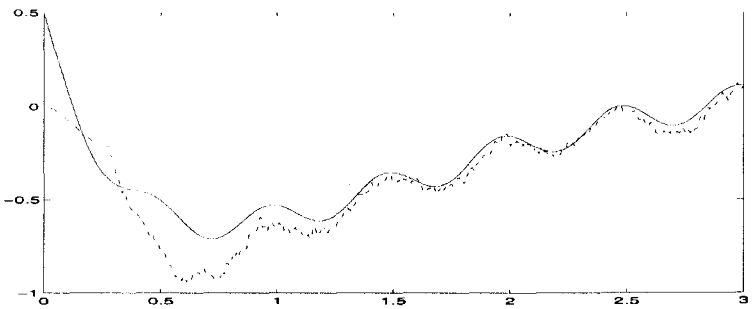

Figure 4.7: x_3 (-) and $\hat{x}_3$ (-.)

4.6 Conclusion

In this chapter, we introduced a sliding observer that does not depend on the derivative of u. This is due to the fact that our main application domain is the AC motor where the derivative of the input does not exist or is not easy to obtain. This appears, for example, when we consider the converter in the observer and control scheme. But, if it exists, and if it is technologically possible to obtain $\dot{u}$, $\ddot{u}$, ..., and so on. A very cleaver observer form was given in [23]. For this form, many observer designs work well, and in this case, advantages of the sliding mode observer were principally the design simplicity and the finite time convergence. In practical observer design, we always take into account the output noise, thus generally we replace the *sgn* function by a modified *sgn* function or higher order sliding mode. In the latter, we think that it is important, when it is possible, as it is proposed in [15], to design an observer without the use of diffeomorphism, because the observer validity domain is restricted to the diffeomorphism validity domain.

4.7 Appendix

4.7.1 Proof of Proposition 39

From Definition 33, the indices μ_i verify:
• $\sum_{i=1}^{p} \mu_i = n$, so from the Definition of $\bar{\mu}_i$, one has: $\sum_{i=1}^{2p} \bar{\mu}_i = n$.

• $\Delta = \left\{ L_f^{j-1}(dh_i) : i = 1, ..., p;\ j = 1, ..., \mu_i \right\}$ are linearly independent.
As $L_f^j(dh_i) = L_f^{j-1}(L_f(dh_i)) = L_f^{j-1}(\dot{y}_i)$. Δ will be rewritten as
$\left\{ L_f^{j-1}(d\bar{y}_i) : i = 1, ..., 2p;\ j = 1, ..., \bar{\mu}_i \right\}$

• Thus, if μ_i verify 3. of Definition 33, it is easy to see that: If $l_1, ..., l_{2p}$ satisfies $\sum_{i=1}^{2p} l_i = n$ and $\left\{ L_f^{j-1}(d\bar{y}_i) : i = 1, ..., 2p;\ j = 1, ..., l_i \right\}$ are linearly independent at some $\xi \in \mathcal{U}$, then $(l_1, ..., l_{2p}) \geq (\bar{\mu}_1, ..., \bar{\mu}_{2p})$ in the lexicographic reordering $[(l_1 > \mu_1)$ *or* $(l_1 = \mu_1$ *and* $l_2 > \mu_2)$ *or... or* $(l_1 = \mu_1, ..., l_p = \bar{\mu}_{2p})]$.

Then, the $2p$-tuple $\bar{\mu}_1, ..., \bar{\mu}_{2p}$ satisfies the three conditions of Definition 33.

4.7.2 Proof of Theorem 41

First, starting from Theorem 36, where y is substituted by $\bar{y}$, one proves hereafter that conditions 1. and 2. of Theorem 35 (which are required in Theorem 36) are equivalent to conditions 1. and 2. of Theorem 41. For the Theorem 36, let us define: $\bar{Q} = \left\{ L_f^{j-1}(d\bar{y}_i) : \ i = 1, ..., 2p; \ j = 1, ..., \bar{\mu}_i \right\}$ and for $j = 1, ..., 2p$ $\bar{Q}_j = \left\{ L_f^{k-1}(d\bar{y}_i) : \begin{array}{l} i = 1, ..., 2p; \\ k = 1, ..., \bar{\mu}_j \end{array} \right\} - \left\{ L_f^{\bar{\mu}_j - 1}(d\bar{y}_j) \right\}$
but
$\bar{Q} = \left\{ L_f^{j-1}(L_f(dh_i)) : \begin{array}{l} i = 1, ..., p; \\ j = 1, ..., \mu_i - 1 \end{array} \right\} \cup \{h_1, ..., h_p\}$ and $L_f^{j-1}(L_f(dh_i)) = L_f^j(dh_i)$, then:

$$\bar{Q} = Q$$

So the equivalence of condition 1. is proved. Now, for condition 2., for Theorem 36 one computes $\bar{Q}_j$.

• For $j = 1, ..., p$ one has:
$\bar{Q}_j = \left\{ L_f^{k-1}\left[L_f(d\bar{y}_i)\right] : \begin{array}{l} i = 1, ..., 2p; \\ k = 1, ..., \bar{\mu}_j - 1 \end{array} \right\}$
and $\bar{y}_j = L_f(h_j)$, so

$$\bar{Q}_j = \left\{ \left[L_f^{k-1}(L_f(dh_i)) : \begin{array}{l} i = 1, .., p; \\ k = 1, .., \mu_j - 1 \end{array} \right] \cup \{h_1, ..., h_p\} \right\} - \left\{ L_f^{\mu_j - 1 - 1}(L_f(dh_j)) \right\}$$

as $L_f^k(L_f(d\bar{y}_{i+p})) = L_f^{k-1}\left[L_f(d\bar{y}_i)\right]$, one immediately has $\bar{Q}_j = Q_j$.

• For $j = p+1, ..., 2p$ one has $\bar{y}_j = h_j$, so
$\bar{Q}_j = \{dh_1, dh_2, .., dh_p, L_f(dh_1), .., L_f(dh_p)\} - \{dh_j\}$ then, as $\mu_i \geq 2$, one obtains $\bar{Q}_j \cap \bar{Q} = \bar{Q}_j$ for $j = 1, ..., p$.
Thus, the condition 2. of Theorem 41 is equivalent to condition 2. of Theorem 35.

Secondly, in the same way, one proves the equivalence between conditions 3. and 4. of Theorem 41, and the last conditions of Theorem 36, where y is substituted by $\bar{y}$. Theorem 36 applied to $\bar{y}$ gives:
There exists a change of coordinates transforming (4.12) into (4.9) if and only if the previous conditions hold and there exist vector fields $\bar{g}^1, \bar{g}^2, ..., \bar{g}^{2p}$ satisfying

$$L_{\bar{g}^i} L_f^{l-1}(\bar{y}_j) = \delta_{i,j}\delta_{l,\bar{\mu}_i}, \quad \begin{array}{l} i, j = 1, ..., 2p, \\ l = 1, ..., \bar{\mu}_i, \end{array}$$

such that

$$[ad^k_{(-f)}\bar{g}^i, ad^l_{(-f)}\bar{g}^j] = 0 \qquad (4.24)$$

for $i, j = 1, ..., 2p$; $k = 0, ..., \bar{\mu}_i - 1$; $l = 0, ..., \bar{\mu}_j - 1$.

Now, one wants to rewrite this condition only as a function of y. Therefore, the p first vector fields $\bar{g}^i$ are defined such that

$$L_{\bar{g}^i} L_f^{l-1}(\bar{y}_j) = \delta_{i,j}\delta_{l,\mu_i-1}, \quad \begin{array}{l} i = 1, ..., p, \; j = 1, ..2p \\ l = 1, ..., \mu_i - 1, \end{array}$$

with the Definition of $\bar{y}_j$, this is equivalent to the real output y to

$$L_{\bar{g}^i} L_f^{l-1}(y_j) = \delta_{i,j}\delta_{l,\mu_i}, \quad \begin{array}{l} i, p = 1, ..., p, \\ l = 1, ..., \mu_i, \end{array} \qquad (4.25)$$

Now, the $p-$last vector fields $\bar{g}^i$ are defined such that

$$L_{\bar{g}^i}(\bar{y}_j) = \delta_{i,j}, \quad \begin{array}{l} i = p+1, ..., 2p, \\ j = 1, ..., 2p \end{array}$$

which can be rewritten as:

$$\begin{array}{ll} L_{\bar{g}^i}(h_j) = \delta_{i,j}, & i = p+1, ..., 2p, \\ L_{\bar{g}^i}(L_f(h_j)) = 0, & j = 1, ..., p \end{array} \qquad (4.26)$$

Thus, from (4.25) and (4.26), one obtains condition 3. of Theorem 41. Moreover, from this and (4.24) one immediately finds condition 4. of Theorem 41 and reciprocally.

4.7.3 Proof of Theorem 49

Sufficiency

If condition 47 holds, then

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} h \\ L_f(h) \\ \vdots \\ l_f^{n-1}h \end{pmatrix}$$

is a diffeomorphism and transforms system (4.15) in

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \xi_2 + \bar{g}_1(\xi, u) \\ \xi_3 + \bar{g}_2(\xi, u) \\ \vdots \\ \xi_n + \bar{g}_{n-1}(\xi, u) \\ \bar{f}_n(\xi) + \bar{g}_n(\xi, u) \end{pmatrix}$$

with $\bar{g}_i(\xi, u = 0) = 0$ for any $i \in \{1, ..., n\}$. Moreover, in the x coordinate, condition 48 is equal to

$$d\bar{g}_i \in span\{dx_1, ..., dx_i\} \quad \forall i \in \{1, n\} \tag{4.27}$$

this implies that the system is in form 4.16.

Necessity
If there exists a diffeomorphism $\xi = \phi(x)$ which transforms (4.15) into (4.16) condition 47 is directly verified by the existence of ϕ. Moreover as (4.27) is a necessary condition, this implies that condition 48 is a necessary condition too.

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