Diagnosis and supervision TSFS06
Exercises

Part I

Exercises

Chapter 1

Introduction to diagnosis

Exercise 1.1.
This exercise is intended for getting acquainted with decisions structures. Observe the following decision structures that describe how test δ_i for $i = 1, 2, 3$ reacts to four behavior modes (consider only single faults):

	NF	F_1	F_2	F_3
δ_1	0	0	X	X
δ_2	0	X	0	X
δ_3	0	X	X	0

a) Which behavior modes should test δ_1 react on?

b) Describe in words the conclusion of test δ_1 when the test does not generate an alarm.

c) Describe in words the conclusion of test δ_1 when the test generates and alarm.

d) Calculate the diagnosis/diagnoses when no test generates an alarm.

e) Calculate the diagnosis/diagnoses when only δ_1 generates an alarm.

f) Calculate the diagnosis/diagnoses when both δ_1 and δ_2 but not δ_3 generate alarms .

Exercise 1.2.
This exercise is intended for providing understanding of how a diagnosis system works. The exercise exemplifies tests and decision structures that are the two fundamental components of a diagnosis system.

Figure 1.1 shows the system that is often called the polybox example. The system consist of five components, the multiplications M1, M2 och M3 and two

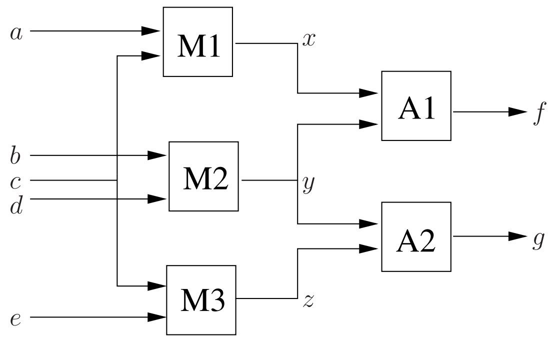


Figure 1.1: The polybox example.

additions A1 och A2. The input a, b, c, d and e and output f and g signal values are known. In this exercise the following modes are considered:

NF	No fault
A1	Arbitrary fault in component A1
A2	Arbitrary fault in component A2
M1	Arbitrary fault in component M1
M2	Arbitrary fault in component M2
M3	Arbitrary fault in component M3

Assume that a diagnosis system has been constructed with the following four test quantities:

$$
\begin{aligned}
T_0 &= |f - ac - bd| + |g - bd - ce| \\
T_1 &= |g - bd - ce| \\
T_2 &= |f - g - ac + ce| \\
T_3 &= |f - ac - bd|
\end{aligned}
$$

The test quantities are compared to a threshold $J = 1$ and are said to react if the value of the test quantity are higher than the threshold. Decisions are specified by the following decision structure:

	NF	A1	A2	M1	M2	M3
T_0	0	X	X	X	X	X
T_1	0	0	X	0	X	X
T_2	0	X	X	X	0	X
T_3	0	X	0	X	X	0

a) Assume a fault in the the addition $A1$ that makes $f = x + y + 2$. With this fault the system outputs are $f = 13$ och $g = 23$ when the inputs are $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$. Given these observations calculate the values of the test quantities. Which tests generate an alarm?

b) Calculate the diagnoses that are given by the test response in the (a) exercise using the decision structure. Comment on whether the result is as expected.

Exercise 1.3.
Model based diagnosis for a process requires a specification of the possible fault that should be diagnosed and a model that describes the behavior of the process for the faults. A common case is that only a model of the fault free behavior is available and that the model must be expanded with information on the behavior of the process when a fault has occurred. The process that should be diagnosed consist of an actuator and two sensors. Assume that the model for a fault free behavior is

$$\begin{aligned} \dot{x} &= u \\ y_1 &= x \\ y_2 &= x \end{aligned} \tag{1.1}$$

where x is unknown, u a known control signal and y_1 and y_2 two sensor signals. Assume that the actuator and the two sensors can fail.

a) List all possible behavior modes. It is sufficient to list all singular faults.

b) Assume that the behavior for a faulty component is unknown. Model this by introducing the fault signals f_1, f_2 and f_3 for the three faults. Indicate which values the fault signals can take for all the listed behavior modes in the answer on the a)-exercise.

Exercise 1.4.
Making a diagnosis requires redundancy. Give examples on static and temporal redundancy in (1.1).

Exercise 1.5. There is redundancy in model (1.1) and therefore residuals can be constructed.

a) Calculate two residuals, one based on static and one on temporal redundancy. You can assume that derivatives of known signals, like e.g. $\dot{y}_1$, are known.

b) For it to be possible to construct a decision structure it is necessary to know which faults that affect each residual. Express the residuals only in the fault signals f_1, f_2 and f_3 introduced in exercise 1.3, to determine which faults the residuals are sensitive to. The expressions for the residuals are then called the internal form of the residuals.

c) Compile the fault sensitivity of the residuals in a decision structure.

Exercise 1.6. Assume that the residuals r_i that were constructed in exercise 1.5 generates alarms when $|r_i| > 1/2$. Use the constructed diagnosis system to calculate the diagnoses when the following values $u = 1$, $y_1 = 0$, $\dot{y}_1 = 0$, $y_2 = 1$ and $\dot{y}_2 = 1$ has been observed.

a) Calculate the value of the residuals and which tests that generate alarms.

b) Use the decision structure to calculate the diagnoses. Assume that only singular faults are considered.

Exercise 1.7.
A rotating system is propelled by a motor. A simple model of the system can be:

$$J\dot{\omega} = -M_{\text{fric}} + M_{\text{motor}}$$

Assume simple viscous friction, i.e. $M_{\text{fric}} = \mu\omega$ where μ is the friction coefficient. Assume also that the motor torque is controlled toward a reference torque u and that the torque control is quick enough for this dynamics to be neglected. The motor torque is then given by $M_{\text{motor}} = ku$, where k is a constant that is 1 when the torque control works. Assume that the process is equipped with two sensors, one that measures the angular velocity ω and one that measures the angular position of the machine φ.

Write the model in state space form och introduce behavior models for the following faults:

1. Increased viscous friction
2. Torque controller malfunction
3. Faults in the angular position sensor
4. Constant bias fault in the angular velocity sensor

Exercise 1.8.
The figure[1] shows an industrial robot IRB1400 from ABB Robotics. A model of the dynamics of the robot around an axle can, with some simplifications, be written as

$$\begin{aligned} J_m\ddot{\varphi}_m &= -F_{v,m}\dot{\varphi}_m + k_T u + \tau_{\text{spring}} \\ \tau_{\text{spring}} &= k(\varphi_a - \varphi_m) + c(\dot{\varphi}_a - \dot{\varphi}_m) \\ J_a\ddot{\varphi}_a &= -\tau_{\text{spring}} \\ y &= \varphi_m \end{aligned}$$

[1] The picture comes from ABBs web page http://www.abb.com/

Symbol	Description
J_m	Angular inertia: motor
J_a	Angular inertia: arm
φ_m	Motor position
φ_a	Arm position
$F_{v,m}$	Viscous friction coefficient, motor
k	Stiffness coefficient, gearbox
c	Damping coefficient, gearbox
k_T	Torque constant, is 1 when the torque control works
u	Torque reference for the torque controller
y	Measured motor position value

Introduce models for the following faults in the model equations:

1. Fault in the torque control for the driving motor.
2. Ground wire for the sensor torn off which causes reduced signal to noise relation in the sensor signal.
3. The robot has a load, attached at the arm tip, that is dropped.
4. The robots arm collides with its surroundings.

Exercise 1.9.
Are there analytic redundancy in the systems described by the following model relations? In the cases analytic redundancy exist, is it static or temporal redundancy?

Variables y_i denotes sensor signals, u control signals, and d_i unknown disturbances.

a)
$$d + y_1 + \dot{y}_2 - u = 0$$

b)
$$\begin{aligned} d + y_1 + y_2 - u &= 0 \\ 2y_1 + d + 3u &= 0 \end{aligned}$$

c)
$$\begin{aligned} d + y_1 + y_2 - u &= 0 \\ 2\dot{y}_1 + d + 3u &= 0 \end{aligned}$$

d)
$$\begin{aligned} \dot{d} + y_1 + y_2 - u &= 0 \\ 2y_1 + d + 3u &= 0 \end{aligned}$$

e)
$$\begin{aligned} d_1 + y_1 + y_2 - u &= 0 \\ 2y_1 + d_2 + 3u &= 0 \end{aligned}$$

Exercise 1.10.
The purpose with this exercise is to understand the concepts of false alarms and missed detection. Consider a residual with the internal form

$$r = f + v \tag{1.2}$$

where f is the fault signal that we want to detect and v is a normal distributed stochastic variable with mean 0 and standard deviation $\sigma = 1$. Based on the residual, a diagnosis test is defined that generates an alarm when $|r| > J$ where $J > 0$ is a predetermined threshold. The conclusions of the test is defined by

	NF	F
	0	X

where NF denotes the fault free mode, i.e. when $f = 0$, and F the fault mode when $f \neq 0$.

a) Make a sketch of residual distribution in the case when $f = 0$ and $f = 3$ and mark a suitable threshold.

b) Define the false alarm probability. The false alarm probability corresponds to an area in the sketch from exercise (a). Mark this area.

c) If X is a normal distributed stochastic variable with mean $\mu = 0$ and standard deviation $\sigma = 1$ then let $\Phi(x)$ be the cumulative distribution function for X, i.e.

$$\Phi(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

where $f(x)$ is the known normal distribution curve

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Using $\Phi(x)$, sketch the probability for false alarms as a function of the threshold J.

d) Define the probability for missed detection given a fault with the size $f = f_0 \neq 0$. Just like the false alarm probability the probability for missed detection corresponds to an area in the sketches from exercise (a). Mark this area.

e) Sketch the probability for missed detection given a fault with the size $f = f_0 \neq 0$ as a function of the threshold J and the cumulative distribution function Φ.

f) What is the best possible false alarms probability and the probability for a missed detection at a test?

Exercise 1.11.
Consider a hypothesis test with the hypotheses

$$\begin{aligned} H_0 &: \theta = 0 \quad \text{fault free} \\ H_1 &: \theta \neq 0 \quad \text{faulty} \end{aligned}$$

and a test quantity T that is written on internal form can be expressed as $T = \theta + v$ where v is a normal distributed stochastic variable with mean 0 och standard deviation σ. The zero hypothesis is rejected if $|T| > J$ where J is a threshold selected so that the probability for false alarm is 10^{-5}, i.e. J is the solution to

$$P(|T| > J|\theta = 0) = 10^{-5}$$

There are two views as to what conclusion that can be drawn when $|T| \leq J$ namely

I) H_0 is true

II) H_0 is not rejected

a) What are the verdict on the parameter θ when $|T| \leq J$ for the hypothesis test in the exercise, according to the two views?

b) What can be said on the probability

$$P(\text{faulty conclusion}|\theta \neq 0)$$

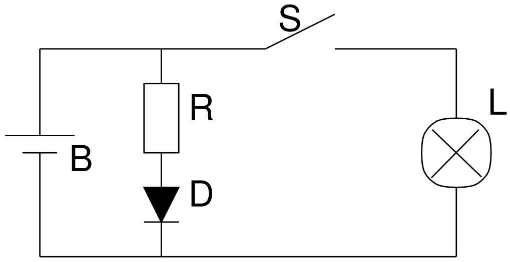
for the two views.

Exercise 1.12.
Consider a SISO-system that is described by the following differential equation:

$$\dot{y} - ay - (b + \Delta b)u = 0$$

where a and b are known nonzero constants, y a sensor signal, u a control signal and Δb a parametrization of a fault in the system. In the normal case $\Delta b = 0$ and when a fault occurs $\Delta b \neq 0$. Assume that the system is controlled towards $y = 0$. Indicate a fundamental problem in detecting changes in the parameter Δb. Do this by analyzing a simplified model that applies under the assumption that the controller works perfectly, i.e. that $y(t) = 0$ for all $t \in \mathbb{R}$. Give a suggestion for how to solve this.

Exercise 1.13.

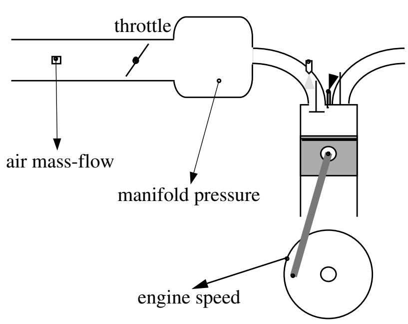


The circuit shown above includes 5 components: a switch (S), a resistor (R), a light emitting diode (D), a battery (B) and a light bulb (L). The following faults are assumed possible in the system: The switch can get stuck in open (SÖ) or closed (SS) position. The light bulb can be broken (LT) and the battery can become discharged (BU). Assume that only singular faults can occur. We can see whether the light bulb and diode are lit and we know the desired position of the switch.

a) for the different combinations of observations (8 st) indicate all possible diagnoses.

b) Which faults can be isolated from each other? Assume that the switch S can be freely controlled.

Exercise 1.14.



Consider the inlet system in the figure above. Let W be the air mass flow, p pressure, y_w the measured air mass flow, y_p the measured pressure and u actuator signal to the throttle. A model for the system is

$$W(t) = f(p(t))\,k(u(t))\,a \tag{1.3}$$

$$y_w(t) = g\,W(t) \tag{1.4}$$

$$y_p(t) = p(t) \tag{1.5}$$

where $g \neq 1$ describes an amplification error in the flow sensor and $a \neq 1$ an offset of the throttle on its axle. Both a and g are assumed to be constants.

a) Write the observation space $\mathcal{O}(NF)$, $\mathcal{O}(F_a)$, and $\mathcal{O}(F_g)$.

b) Can the two faults be isolated from each other?

Chapter 2

Fault isolation

Exercise 2.1.
The exercise is intended for understanding how a decision structure is set up given a number of tests and their fault sensitivity. Study the examples in section 3.4.6 of the compendium.

a) Derive the decision structure for the diagnosis system that is described in the example which contains equation (3.18) in section 3.4.6.

b) Derive the decision structure for the diagnosis system that is described in the example which contains equations (3.19) and (3.20) in section 3.4.6.

Exercise 2.2.
This is an exercise of how to calculate diagnoses given a decision structure and a number of tests that has generated alarms. Consider the decision structure

	NF	F_1	F_2	F_3
T_1	0	X	0	X
T_2	0	1	X	0
T_3	0	0	X	0

Assume that $T_i > J_i$ means reject H_i^0 for $i \in \{1, 2, 3\}$.

a) Determine the decisions that are taken when each test generates an alarm or not, i.e indicate S_i^0 and S_i^1 for $i = 1, 2, 3$.

b) What is S_i, $i = 1, 2, 3$, when $T_1 < J_1$, $T_2 > J_2$, $T_3 > J_3$. Show also the calculation of S.

c) What is S_i, $i = 1, 2, 3$, when $T_1 > J_1$, $T_2 < J_2$, $T_3 < J_3$. Show also the calculation of S.

Exercise 2.3.

I	f_1	f_2	f_3
T_1	1	1	0
T_2	1	0	1
T_3	1	1	1

II	f_1	f_2	f_3
T_1	1	1	0
T_2	X	0	1
T_3	1	X	1

State the conclusion for both of the decision structures above when only the following quantities are significantly separate from 0:

a) T_2 and T_3

b) T_1 and T_3

c) T_1

d) T_1 and T_2

e) The only difference between the decision structures is that some 1:s in case I have been switched to X:s in case II. Compare the diagnoses for case I and II in subproblems (a)-(d). Is there any relation between the diagnoses calculated with decision structure I and the diagnoses calculated with decision structure II?

Exercise 2.4.
Consider the decision structure

	NF	F_1	F_2	F_3
T_1	0	0	X	0
T_2	0	0	X	1
T_3	0	X	0	X

Assume that the fault mode F_2 occur and that $T_1 > J_1$, $T_2 > J_2$ and $T_3 < J_3$. Verify that F_2 is isolated uniquely, i.e. $S = \{F_2\}$. What happens with the diagnosis statement S if a disturbance affects the system so that $T_3 > J_3$?

Exercise 2.5.
This exercise intends to provide understanding of the concepts of detectability and isolability for a given diagnosis system. consider a diagnosis system that consists of the residuals

$$r_1 = f_1 - f_2$$
$$r_2 = -f_1 - f_3$$

that are here given on internal form and the decision structure

	NF	f_1	f_2	f_3	f_4
r_1	0	X	X	0	0
r_2	0	X	0	X	0

Assume that a decision is taken if $r_i \neq 0$.

a) Describe what is meant by saying that a fault f_i is detectable with the given diagnosis system.

b) Show that f_1 is detectable with the given diagnosis system.

c) Which faults are detectable?

d) Describe what is meant by saying that a fault f_i is isolable from a fault $f_j \neq f_i$ with the given diagnosis system.

e) Which faults are f_2 isolable from in the exemple? Answer the question by assuming a fault in f_2, identify which residuals react and finally calculate the diagnosis/diagnoses for this test response.

f) Compile the detectability and isolability for the diagnosis system.

Exercise 2.6. Consider the model

$$\begin{aligned} \dot{x} &= u + f_3 \\ y_1 &= x + f_1 \\ y_2 &= x + f_2 \end{aligned} \tag{2.1}$$

where u, y_1 and y_2 are known signals, x an unknown and f_1, f_2 and f_3 fault signals.

a) Define that a fault f_i is detectable in a model. Which faults are detectable in the model?

b) Define that a fault f_i is isolable from a fault $f_j \neq f_i$ in a model. Which singular fault isolability does the model give?

c) assume that a diagnosis system has been constructed with the residuals

$$\begin{aligned} r_1 &= y_1 - y_2 \\ r_2 &= u - \dot{y}_1 \end{aligned}$$

and the decision structure

	NF	f_1	f_2	f_3
r_1	0	X	X	0
r_2	0	X	0	X

State the diagnosis systems detectability and isolability. How does the diagnosis systems detectability and isolability differ from that of the model?

d) For the diagnosis system to achieve the same detectability and isolability as the model it is necessary to add another residual to the two already available residuals. What fault sensitivity must this residual have?

e) Derive a residual with the desired fault sensitivity. You may assume that derivatives of known signals are known.

Exercise 2.7.
The goal in this exercise is to gain understanding of how tests should be added for achieving a specified detectability and isolability. Assume a system with three singular faults f_1, f_2 and f_3. A diagnosis system should be constructed

so that all faults are detectable and the following isolability is achieved

	f_1	f_2	f_3
f_1	X	0	0
f_2	0	X	X
f_3	0	X	X

a) assume that the model enables that a maximum of two faults can be decoupled in each residual. Calculate a decision structure with a minimum of number of rows that fulfills the detectability and isolability specifications. What detectability and isolability gives the designed decision structure?

b) Do the same exercise as in (a) but assume that only one fault at the time can be decoupled.

Exercise 2.8.
Consider a system that can be modeled according to

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u + f_u) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d$$
$$y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where d is a nonmeasurable disturbance and f_u, f_1 and f_2 are three fault signals. Four behavior models are considered:

NF	$\theta = [0\ 0\ 0]$
F_u	$\theta = [f_u(t)\ 0\ 0],\quad f_u(t) \not\equiv 0$
F_1	$\theta = [0\ f_1(t)\ 0],\quad f_1(t) \not\equiv 0$
F_2	$\theta = [0\ 0\ f_2(t)],\quad f_2(t) \not\equiv 0$

It is desired that the diagnosis system uses the following four hypothesis tests:

$$H_1^0 : \mathbf{F_p} \in M_0 = \{\mathbf{NF}\} \qquad H_1^1 : \mathbf{F_p} \in M_0^C = \{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_u}\}$$
$$H_2^0 : \mathbf{F_p} \in M_1 = \{\mathbf{NF}, \mathbf{F_u}\} \qquad H_2^1 : \mathbf{F_p} \in M_1^C = \{\mathbf{F_1}, \mathbf{F_2}\}$$
$$H_3^0 : \mathbf{F_p} \in M_2 = \{\mathbf{NF}, \mathbf{F_1}\} \qquad H_3^1 : \mathbf{F_p} \in M_2^C = \{\mathbf{F_u}, \mathbf{F_2}\}$$
$$H_4^0 : \mathbf{F_p} \in M_3 = \{\mathbf{NF}, \mathbf{F_2}\} \qquad H_4^1 : \mathbf{F_p} \in M_3^C = \{\mathbf{F_u}, \mathbf{F_1}\}$$

For each test quantity (i.e. residual generator) that should be constructed, indicate which signals that need to be decoupled.

Exercise 2.9.
Assume a decision structure according to:

	f_1	f_2	f_3
r	X	0	X

Why are disturbances and faults f_2 equivalent, as seen from the residual r?

Exercise 2.10.

Consider a time discrete system that can be modeled according to

$$
\begin{aligned}
x(t+1) &= (a + \Delta a(t))x(t) + b(u(t) + f_u(t)) \\
y(t) &= x(t)
\end{aligned}
$$

where a and b are known parameters, $y(t)$ and $u(t)$ are known scalar signals, $f_u(t)$ and $\Delta a(t)$ are two fault signals and $x(t)$ an unknown scalar signal.

a) Assume that the parametric fault Δa is modeled with an arbitrary additive signal $f_a(t)$, i.e. $f_a(t) = \Delta a(t)x(t)$. Can f_a and f_u be isolated? If that is the case, construct two test quantities with which f_a and f_u can be isolated.

b) Assume that we are modeling the parametric fault Δa so that Δa is assumed to be constant, i.e. $f_a(t) = \Delta a x(t)$. Can f_a and f_u be isolated? If that is the case, construct two test quantities with which f_a and f_u can be isolated.

c) Assume that we are modeling the parametric fault Δa so that Δa is assumed to be constant, i.e. $f_a(t) = \Delta a x(t)$. Assume also that $f_u(t)$ is constant, i.e. $f_u(t) \equiv c_u$. Can f_a and f_u be isolated? If that is the case, construct two test quantities with which f_a and f_u can be isolated.

Exercise 2.11.

Prove that a diagnosis system that always follows the rules (3.12) always gives a ”complete diagnosis statement”.

Chapter 3
Design of Test Quantities

Exercise 3.1.
Assume that a residual generator have been constructed with the following internal form

$$r_{\text{intern}} = f + v$$

where f is the fault signal that we want to detect and v is a normal distributed stochastic variable with mean value of 0 and a standard deviation of σ. Based on the residual define a diagnosis test that is triggered when $|r| > J$ where $J > 0$ is a predetermined threshold.

a) Define and illustrate the probability of false alarm and the probability of missed detection for a fault of size $f = f_0 \neq 0$. The probabilities can be illustrated in a figure that shows the distribution of the residual with a given value on the threshold. Sketch the figure and highlight the probabilities in an appropriate way.

b) Let $\Phi(x)$ and $\Gamma(p)$ be defined as

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds, \quad \Gamma(z) = \Phi^{-1}(z)$$

and then the following holds

$$P(X \leq x) = \Phi(x), \text{ for } X \sim \mathcal{N}(0, 1)$$

For the test that has been defined above, illustrate with the help of $\Phi(x)$ and $\Gamma(z)$

1. probability of false alarm as a function of the threshold J
2. probability of missed detection given a fault of size $f = f_0 \neq 0$ as a function of the threshold J.

3. how the threshold J can be calculated given a value α on the probability of false alarm.

c) Describe, with words or with a figure, how the probability of false alarm and the probability of missed detection depends on the choice of the threshold J?

d) Define the power function, draw the typical shape of it and mark the probabilities for false alarm and missed detection in the figure. Also state the power function using $\Phi(x)$ and $\Gamma(z)$ for the situation in this exercise.

Exercise 3.2. (D)
Consider the same residual as in Exercise 3.1 and let $\sigma = 2$.

a) Calculate a numerical value on the threshold J such that the probability of false alarm becomes $\alpha = 0.01$.

b) Calculate the power function for the calculated threshold value in Exercise (a). Verify using the power function that the probability of false alarm is $\alpha = 0.01$.

c) For a given fault size, for example $f_0 = 5$, it can be interesting to study how the threshold influences the compromise between achieving low probability of false alarm, p_{fa}, and high probability of detection p_d. Investigate the compromise by draw p_d as a function of p_{fa} and interpret the results.

Exercise 3.3. (D)
During launches of manned space shuttles it has been shown that a fault in the fuel system leads to almost an immediate explosion of the shuttle. The probability that the fault will appear during a launch has been estimated to be $p_1 = 0.01$, which is considerably more often than is tolerated since the crew is killed as a result of the explosion. It is impossible to avoid the fault by the means of re-designing the space shuttle but analyzes of the system have shown that it is possible to detect the fault by the following residual

$$\begin{aligned} r &= v && \text{no fault} \\ r &= f + v && \text{fault} \end{aligned}$$

where $v \backsim N(0, \sigma = 2)$ and $f = 5$. If the fault is detected, that is if $r > J$ for any given threshold J, then the crew can be ejected from the space shuttle. The probability that the escape from the space shuttle leads to the death of the crew is estimated to $p_2 = 0.005$. Calculate the threshold J that minimizes the probability of the death of the crew. Becomes it safer by introducing the test and, if yes, how much safer? The events that the crew will be killed during an escape and that a fault will appear during the launch is assumed to be independent of each other. It is sufficient to calculate a numerical value on the threshold J and a possible safety gain.

Exercise 3.4.
Consider a system that can be modeled with a linear regression model

$$y(t) = u(t)\theta + v(t)$$

where $y(t)$ and $u(t)$ are known, θ describes the fault states and $v(t) \backsim N(0, \sigma^2)$ is white normal distributed noise. Consider the behavioral modes NF (No Fault), F_1, F_2, and F_3. The fault states $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$ for the behavioral modes are

$$\begin{aligned} \Theta_{NF} &= \{[0,0,0]^T\} \\ \Theta_{F_1} &= \{[\theta_1,0,0]^T; \theta_1 \neq 0\} \\ \Theta_{F_2} &= \{[0,\theta_2,0]^T; \theta_2 \neq 0\} \\ \Theta_{F_3} &= \{[0,0,\theta_3]^T; \theta_3 \neq 0\} \end{aligned}$$

Assume that $y(1), u(1), y(2), u(2), \dots y(N), u(N)$ have been observed and the fault state can be assumes to be constant. A diagnosis system shall be constructed with three tests. For each test specify a test quantity, the distribution of the test quantity given the null hypothesis is true, and also how the threshold for the test quantity can be formulated. The threshold shall be set so the probability for false alarm becomes p_{fa}.

a) Construct a test quantity using the methodology from Section 4.2, p. 111 in compendium, ”Test Quantities Based on Prediction Errors”, that corresponds to the following hypothesis test:

$$H_0 : F_p \in \{NF, F_1\} \qquad H_1 : F_p \in \{F_2, F_3\}$$

Hint: If $y(t) = \varphi(t)\theta_0 + v(t)$ where $v(t)$ are independent stochastic variables with a distribution $v(t) \sim N(0, \sigma^2)$ and m is the number of parameters in θ the following holds

$$\min_{\theta} \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \varphi(t)\theta)^2 \sim \chi^2(N-m)$$

where $\chi^2(N-m)$ represents a χ^2-distribution with $N-m$ degrees of freedom.

b) Construct a test quantity with the log-likelihood function (Section 4.4, page 124 in the compendium) which represents a row of

	NF	F_1	F_2	F_3
T	0	X	0	0

in the decision structure. Take advantage of that the noise is normal distributed and simplify as much you can.

c) Construct a test quantity using estimates of the parameters (Section 4.5, p. 126 in the compendium) that decouples the behavioral mode F_1 and F_3. The threshold and distribution does not need to be calculates for this test quantity since this is done in Exercise 3.6.

d) State the decision structure for all these three tests.

Exercise 3.5.
This exercise repeat some fundamental properties of linear regression. Assume a model according to

$$y(t) = au(t) + b + v(t) \tag{3.1}$$

where a and b are constant parameters and $v(t)$ white noise which distribution of the amplitude is $N(0, \sigma_v^2)$.

a) A linear regression can be written on matrix form as

$$Y = \Phi\theta + V \tag{3.2}$$

where Y is a known column vector, Φ is a known matrix with the same amount of columns as unknown parameters in the column vector θ and V is a column vector with independent and uniformly distributed stochastic variables. Write the matrices in the regression for (3.1) when a and b will be estimated from the data $y(t)$, for $t = 1, \ldots, N$ and $u(t)$ for $t = 1, \ldots, N$.

b) Show that for the model (3.2) the estimation

$$\hat{\theta} = \arg\min_{\theta} \sum_{t=1}^{n} (y(t) - \hat{y}(t|\theta))^2$$

is given by the following expression

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T Y \tag{3.3}$$

provided that we have enough excitation. Show that the excitation here means that the $u(t)$ cannot be constant.

c) Show that if $V \sim N(0, I\,\sigma^2)$ then $\hat{\theta} \sim N(\theta, (\Phi^T\Phi)^{-1}\sigma_v^2)$.

Hint: The covariance matrix of a stochastic vector with mean value of 0 is defined as $\text{cov}(V) = E\{VV^T\}$. Use this to first decide which distribution $T = KV$ have where K is an arbitrary constant matrix.

Exercise 3.6.
This exercise gives an understanding in how linear regression can be used to compute test quantities according to the method for parameter estimation and the method for prediction error when the underlying model is linear with unknown parameters.

a) Show how test quantity from the method of prediction error in Exercise 3.4 (a) can be written as a linear regression.

b) Same as (a) but using the test quantity from Exercise 3.4 (b) instead.

c) Show how the test quantity from the method of parameter estimation in Exercise 3.4 (c) can be written as a linear regression. Specify also the distribution of the test quantity when the null hypothesis is true and how the threshold can be formulated when the probability for false alarm shall be p_{fa}.

Exercise 3.7. (D)
The diagnosis system developed in Exercise 3.4 and 3.6 will be implemented in the skeleton file `Batch_Tests/testquantities.m`. Data are generated by the file `Batch_Tests/GenerateTestData.m`.

a) Implement the three test quantities. Write the code in such a way that they become non-negative and normalize them such that $P(T_i > 1) = p_{fa}$. Let the probability of false alarms be $p_{fa} = 1\%$, the standard deviation of the noise is $\sigma = 1$ and the number of samples that a test is based on is $N = 100$.

b) Evaluate if the error sensitivity is the one that was expected by estimate $P(T_i > 1|m)$ for all $i \in \{1, 2, 3\}$ and $m \in \{NF, F_1, F_2, F_3\}$ by Monte-Carlo-simulations. In the file `testquantities.m` the number of realization for every mode m of M is set to $M = 1000$. Further, the size of the fault was set to 1, that is, $\theta_i = 1$ when $m = F_i$.

Exercise 3.8.
Consider the same model as in Exercise 3.4. A test quantity constructed according to the prediction error method is

$$T = \underset{\hat{\theta}_1}{\arg} \min_{\hat{\theta}_1, \hat{\theta}_2} \sum_{t=1}^{N} (y(t) - \hat{y}(t|\,\theta = [\hat{\theta}_1\, \hat{\theta}_2\, 0]'))^2$$

Assume that

$$U = \begin{bmatrix} u(1)^T & u(2)^T & \dots & u(N)^T \end{bmatrix}^T \tag{3.4}$$

has full column rank.

a) Which faults are decoupled in the test quantity?

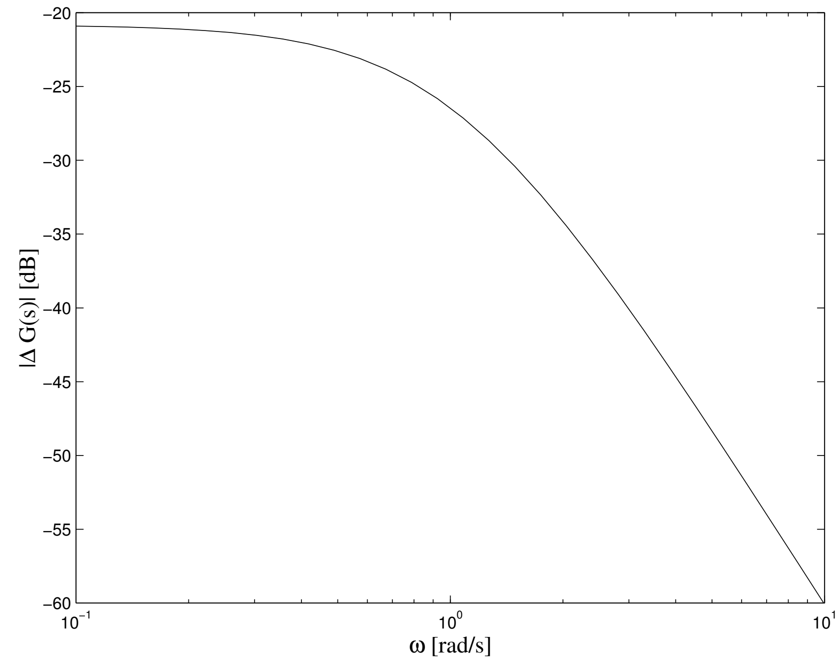
b) Which hypotheses do the test quantity tests?

c) Specify how the row in the decision structure corresponding to T looks like.

Exercise 3.9.
Assume a model as

$$y = (G(s) + \Delta G(s))u + L(s)f$$

where $G(s) = \frac{1}{s+1}$ and $L(s) = 1$. Additionally there is a estimation on the upper boundary for $|\Delta G(j\omega)|$ according to

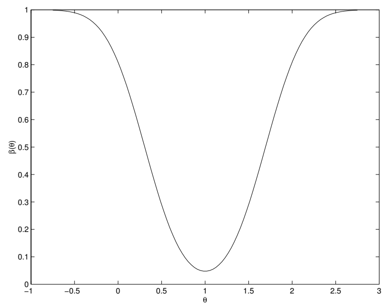


Construct a residual generator and an adaptive threshold based on this information.

Exercise 3.10.
Consider a diagnosis system with only one hypothesis test. The test quantity for the hypothesis test with the following hypothesis

$$H_0 : \theta = 1$$
$$H_1 : \theta \neq 1$$

has the power function $\beta(\theta)$ which is shown below.



a) How can the power function be used to determine the probability for false alarm $P(FA)$?

b) Given fault corresponding to $\theta \neq 1$ what is the probability of missed detection $P(MD|\theta)$?

Exercise 3.11.

a) Assume the linear and discrete system as

$$x(t+1) = Ax(t) + Bn(t)$$

where A is stable and n is white noise with a noise intensity of Σ_n, that is

$$E\{n(t)n^T(t-\tau)\} = \Sigma_n \delta(\tau)$$

where $\delta(\tau)$ is the discrete dirac delta function. Show that the covariance of vector x is given by the symmetrical solution to the (Lyapunov-)equation

$$\Sigma_x = A\Sigma_x A^T + B\Sigma_n B^T$$

Hint: Start with the definition $\Sigma_x = E\{x(t)x(t)^T\}$.

b) Using the result in Exercise (a), formulate the covariance of the output signal y for the system

$$\begin{aligned} x(t+1) &= Ax(t) + Bn(t) \\ y(t) &= Cx(t) \end{aligned}$$

c) What happens in Exercise (b) if there exists a direct term, that is

$$\begin{aligned} x(t+1) &= Ax(t) + Bn(t) \\ y(t) &= Cx(t) + Dn(t) \end{aligned}$$

⋆ d) Assume a linear time-continuous system as

$$\dot{x} = Ax + Bn$$

where A is stable and n is white noise with a noise intensity of Σ_n, that is

$$E\{n(t)n^T(t-\tau)\} = \Sigma_n \delta(\tau)$$

where $\delta(\tau)$ is the time-continuous dirac delta function. Show that the covariance of the vector x is given by the symmetrical solution to the (Lyapunov-)equation

$$A\Sigma_x + \Sigma_x A^T + B\Sigma_n B^T = 0$$

3 hints:

- $\Sigma_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega) d\omega$ where $\Phi_x(\omega)$ is x spectrum.
- $\mathcal{F}^{-1}\{(j\omega I - A)^{-1}\} = e^{At}$ if A is stable
- Parseval theorem and integration by parts

Exercise 3.12.
Assume that $v(t)$ is white noise with expectation of 0 and variance σ_v^2. What can be said about

a) Auto-correlation (also called function of covariance) of $v(t)$, that is

$$r_v(k) = E\{v(t)v(t-k)\}$$

b) The dependency between $v(t)$ and $v(t-k)$ for $k \neq 0$?

Exercise 3.13. (D)

a) Assume a test quantity T with distribution $T \in N(\theta, \sigma)$, where

$$\sigma = 0.7$$

The null hypothesis is that $\theta = 0$. Determine the threshold in such a way that the significance of the test becomes $\alpha = 0.05$. The system alarms when $|T|$ exceeds the threshold.

b) Calculate the power function $\beta(\theta)$ given the threshold from Exercise-(a). You should do this both by analytical calculations and Monte-Carlo simulations.

Exercise 3.14. (D)

a) Let θ be a two-dimensional vector and $T = |\hat{\theta}|$ where $\hat{\theta} \in N(\theta, \Sigma)$ with covariance matrix

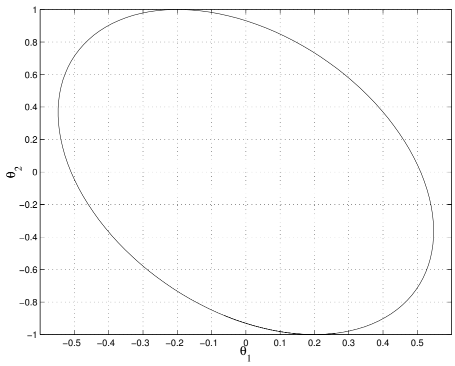
$$\Sigma = \begin{pmatrix} 0.3 & -0.2 \\ -0.2 & 1 \end{pmatrix}$$

Under the null hypothesis the following holds $\theta = 0$. Given a threshold $J = 1.372$ calculate the power function along the axis of θ_1 and θ_2, that is, calculate $\beta(\theta_1)$ with $\theta_2 = 0$ and $\beta(\theta_2)$ with $\theta_1 = 0$.

These calculations are preferable executed using Monte-Carlo simulations. Interpret the results.

b) The covariance matrix have non-zero element outside the diagonally, that is, the estimation of θ_1 and θ_2 are correlated. The following is true $E(\hat{\theta}_1\hat{\theta}_2) = -0.2$.

The areas of significance becomes in this specific case not circles but ellipses according to



What does this cross-covariances *means*? How do this distortion affect the value of the threshold?

c) It is possible to find a K such that $K\hat{\theta}$ has a diagonal in the covariance matrix. One possible test quantity is $T = \sigma^{-2}|K\hat{\theta}|^2 \in \chi^2(2)$ where σ^2 is the variance for the two estimations in $K\hat{\theta}$. What is the null hypothesis of the test? Is it possible with this test to isolate faults $\theta_1 \neq 0$ from $\theta_2 \neq 0$ and vice versa?

$\star$ d) Calculate a K such that KT has a diagonal covariance matrix. Hint: Use a symmetrical and positive semidefinite matrix Σ which can be written as

$$\Sigma = USU^T$$

where S is a diagonal matrix with Σ:s eigenvalues in the diagonally and U is an orthogonal matrix. See `svd` in Matlab.

Exercise 3.15. (D)
Consider the system

$$y(t) = \theta + v(t)$$

where v is white noise with a distribution of its amplitude as $N(0, \sigma)$. Assume that given N samples of y shall test the following hypotheses

$$H_0 : \theta = 0 \qquad H_1 : \theta \neq 0$$

using the test quantity

$$T(y) = \sum_{t=1}^{N} y(t)^2$$

Under the null hypothesis the following holds $T(y)/\sigma^2 \in \chi^2(N)$ but when $\theta \neq 0$ the expression gets messy.

Estimate, using Monte-Carlo simulations the distribution of $T(y)$ for different θ.

Exercise 3.16.
Assume a test with the following hypotheses

$$H_0 : \theta = 0 \qquad H_1 : \theta = 1$$

where θ either has the value 0 or 1. The test quantity $T(x)$ has the distribution $N(\theta, 0.15)$ and the threshold is 0.5. This corresponds to that the probability of false alarm α is less than 0.0005.

Assume a distribution of θ according to

$$p(\theta) = \begin{cases} 0.9999 & \theta = 0 \\ 0.0001 & \theta = 1 \end{cases}$$

that is, with high probability the null hypothesis is true. In the diagnosis application this is fair to assume since the null hypothesis often includes the fault free case which (hopefully) is the most likely behavioral mode.

a) Calculate the probability that given H_0 is true that the H_0 will be rejected, that is

$$P(H_0 \text{ true} | 0.5 < T(x))$$

b) Assume there exists two *independent* samples x_1, x_2 that are available. Assume also that H_0 only is rejected if both $T(x_1)$ and $T(x_2)$ are above their thresholds. Now, calculate the probability that H_0 is true given that H_0 is rejected, that is

$$P(H_0 \text{ true}|0.5 < T(x_1) \wedge 0.5 < T(x_2))$$

c) Calculate how many independent samples are needed to be able to

$$P(H_0 \text{ true}|0.5 < T(x_1) \wedge \cdots \wedge 0.5 < T(x_n)) < \alpha$$

Exercise 3.17.
This exercise points out that estimation by linear regression is not unbiased when the noise is colored.

Assume that the model structure is the following:

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) \\ y(t) &= x(t) + v(t) \end{aligned}$$

that is, the white noise is introduced as measurement noise in a dynamic system.

a) Write the above statements on the form $y(t|t-1) = \varphi(t)\theta + w(t)$. With $y(t|t-1)$ denoting the one-step-prediction, that is, $y(t-1)$ and $u(t-1)$ can be a part of $\varphi(t)$.

b) Are $w(t)$ and $w(t-1)$ uncorrelated?

c) Write $E\{\hat{\theta}\}$ when $\hat{\theta}$ is estimated with the least square method $\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y$. Please comment the results.

Exercise 3.18. (D)
Consider a model $y(t) = b_1u(t-1) + b_2u(t-2) + v(t)$, where $v(t) \in N(0, 0.1)$ is uncorrelated noise. Re-write the model as a linear regression and perform a least squares estimation of b_1 and b_2

a) With data from `PersistentExcitation/data_a.mat`.

b) With data from `data_b.mat`.

c) Is there any differences between the estimation? If yes, why?

d) How can this be avoided?

e) Discuss the difference on the data in Exercise (a) and (b) if the test quantity would be $y - \hat{y}$ or $\hat{\theta} - \theta_{\text{nominal}}$.

Exercise 3.19. (D)
A skeleton file for this exercise is available at `CUSUM_GLR/cusum_glr.m`

a) Load the data from file `CUSUM_GLR/cusum_a.mat`. The signal x is white noise with an amplitude distribution of

$$X(t) \in \begin{cases} N(0,1) & t < t_{ch} \\ N(0.3,1) & t \geq t_{ch} \end{cases}$$

Use the CUSUM algorithm from Section 4.7 in the compendium, alternative Section 2.2 in the distributed material, to get t_{ch}. Compare the results with your own estimate from visual review of the signal.

b) Load the data from file `cusum_b.mat`. The signal x is white noise with an amplitude distribution of

$$X(t) \in \begin{cases} N(0,1) & t < t_{ch} \\ N(0,1.2^2) & t \geq t_{ch} \end{cases}$$

that is, a change in variance compared in the expectation value. Use once again the CUSUM algorithm to get t_{ch}.

c) Assume that in Exercise-(a) the expectation value after t_{ch} is not known. Use the GLR algorithm to compute an estimate of t_{ch}. See Section 2.4 (more specific Section 2.4.3) in the extra material from the book "Detection of Abrupt Changes" by M. Basseville, I.Nikiforov. Compare the performance compared to the Exercise-(a).

d) Solve the detection problem from Exercise-(c) with a simple low-pass filter with a constant threshold. Discuss the differences.

Exercise 3.20. (D)
The CUSUM-algorithm is derived, as shown in the compendium and the extra distributed material, formally under the assumption that the distribution of the residual is known, both in the fault free case and also when a fault has occurred. This is often not a realistic situation and this exercise tries to illustrate how the CUSUM-algorithm can be useful even if detailed statistical knowledge is missing.

A skeleton file is available at `CUSUM_Res/cusumres.m` and data for this exercise can be loaded from the file `cusumres.mat`. In that file there is a vector of time, t, control and measurement signals $y(t)$ and $u(t)$, respectively, and also a model of the process $G(q)$ for the scalar system.

a) Construct a residual generator for detection of faults in the sensor. Choose the parameters for the filter in such a way that noise is reduced to a moderate level and set a threshold.

b) Redo the Exercise-(a), but this time use the CUSUM-algorithm instead. Use the algorithm to, in case of an alarm, estimate the time the fault occurred.

c) Discuss the differences between the two solutions in (a) and (b).

Exercise 3.21. (D)
This exercise includes a construction of a small diagnosis system. Three behavioral modes are treated NF (no fault), MC (mean change), and DC (standard deviation change). The model of the system can be described as

$$x(t) = \begin{cases} v(t) & \text{if } t < t_{ch} \\ \sigma\, v(t) + \mu & \text{if } t \geq t_{ch} \end{cases}$$

where $v(t) \in N(0,1)$, $v(i)$ and $v(j)$ are independent for $i \neq j$. The behavioral mode NF means that $\sigma = 1$ and $\mu = 0$. The behavioral mode MC means that $\sigma = 1$ and $\mu \neq 0$. Finally, the behavioral mode DC means that $\sigma \neq 1$ and $\mu = 0$. A skeleton file in Matlab for a diagnosis system is available in the file `diagsys.m`. That file and other files that are needed for this exercise is available at the following library `MLR_Diag/`.

In those cases where optimization is needed there is no requirements that the implementation needs to be efficient. Exhaustive searches are OK.

a) Express θ and determine the following sets Θ_{NF}, Θ_{MC}, and Θ_{DC}.

b) The diagnosis system contains three tests of the hypotheses. The sets M_i are $M_1 = \{NF\}$, $M_2 = \{NF, MC\}$, and $M_3 = \{NF, DC\}$. Examine the code in the file `diagsys.m` and write correct conclusions (diagnosis statements) S_i for every test. How do the decision structure looks like?

c) The Maximum Likelihood Ratios for the three tests are:

$$\begin{aligned}
\lambda_1(X) =& \frac{\sup_{\theta \in \Theta_{MC} \cup \Theta_{DC}} L(\theta, X)}{L(\theta_0, X)} = \\
=& \frac{\max(\sup_{\theta \in \Theta_{MC}} L(\theta, X), \sup_{\theta \in \Theta_{DC}} L(\theta, X))}{L(\theta_0, X)} \approx \\
\approx& \frac{\max(\sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} L(\theta, X), \sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} L(\theta, X))}{L(\theta_0, X)} = T_1(X) \\
\lambda_2(X) =& \frac{\sup_{\theta \in \Theta_{DC}} L(\theta, X)}{\sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} L(\theta, X)} \approx \frac{\sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} L(\theta, X)}{\sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} L(\theta, X)} = T_2(X) \\
\lambda_3(X) =& \frac{\sup_{\theta \in \Theta_{MC}} L(\theta, X)}{\sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} L(\theta, X)} \approx \frac{\sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} L(\theta, X)}{\sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} L(\theta, X)} = T_3(X)
\end{aligned}$$

The expressions above defines the test quantities $T_i(X)$. How come is the approximation very good?

d) Show that

$$\ln T_2(X) = \sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} \ln L(\theta, X) - \sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} \ln L(\theta, X)$$

e) In the code we use $\ln T_i(X)$ instead of $T_i(X)$. Thus, for calculation of the three test quantities we need expressions of

$$\begin{aligned}
\sup_{\theta \in \Theta_{NF} \cup \Theta_{MC}} \ln L(\theta, X) &= \texttt{lnLmu(X)} \\
\sup_{\theta \in \Theta_{NF} \cup \Theta_{DC}} \ln L(\theta, X) &= \texttt{lnLstd(X)} \\
\ln L(\theta_0, X) &= \texttt{lnL0(X)}
\end{aligned}$$

which also defines the three functions `lnLmu(X)`, `lnLstd(X)`, and `lnL0(X)`.

The code for `lnLmu(X)` is already implemented. Now, implement also the code for `lnL0(X)`.

f) Test the implementation of `lnL0(X)` and `lnLmu(X)` with data generated in the same way as in the file `datagen.m`.

g) Implement `lnLstd(X)` in the code and test the diagnosis system with data generated in the same way as in the file `datagen.m`.

h) Use the diagnosis system to get diagnosis from the data given in the following files `x1`, `x2`, `x3`, `x4`, and `x5`. What are the conclusions from the diagnosis system for these signals? In those cases that the diagnosis detects a fault, when are the faults occurring and what are the sizes of the faults?

Exercise 3.22.
Consider the system that was developed in Exercise 3.21. Construct an ML (Maximum Likelihood) estimator for all unknown variables t_{ch}, μ, and σ. Run the estimator on data from the files `x1`, `x2`, `x3`, `x4`, and `x5`. What are the conclusions? Compare the performance with the system that was constructed in Exercise 3.21. Discuss the differences and assumptions made in both of the approaches.

Exercise 3.23.
In this exercise we will use the same notation as in the material from "Detection of Abrupt Changes".

The density function of y is normal distributed with mean value μ and standard deviation σ

$$p_\theta(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

The probability for a given value y_k from this distribution is given by

$$p(y_k|\theta) = p_\theta(y_k)$$

a) Consider Example 2.1.1 in "Detection of Abrupt Changes" where the mean value is changed from μ_0 to μ_1. *Show* "log-likelihood" ratio according to the Equation (2.1.7) given Equation (2.1.5), that is, show

$$s_i = \frac{b}{\sigma}\left(y_i - \mu_0 - \frac{\nu}{2}\right)$$

where

$$\nu = \mu_1 - \mu_0$$
$$b = \frac{\mu_1 - \mu_0}{\sigma}$$

b) What does it mean that $s_i > 0$? Also, what does it mean that $s_i < 0$?

c) Consider Equation (2.1.2). What is the indication if S_j^k is larger than all others $S_{j_i}^k$ where $j_i \neq j$?

d) Consider Example (2.1.3) where the standard deviation is changed from σ_0 to σ_1. *Show* that Equation (2.1.23) is correct if y is normal distributed.

$$s_k = \ln \frac{\sigma_0}{\sigma_1} + \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right) \frac{(y_k - \mu)^2}{2}$$

e) (GLR) Consider Example (2.4.3) where the mean value before the change is μ_0 and the mean value after the change is unknown. Equation (2.4.37) gives g_k. Show if $v_m = 0$ then Equation (2.4.40) can be derived from Equation (2.4.37).

Chapter 4

Linear Residual Generation

Exercise 4.1.
Assume that a model of the supervised process is

$$\begin{aligned} \dot{x} &= -ax + u \\ y &= x + f \end{aligned}$$

where y and u are known measurement and control signals, respectively, the signal f is modeling a fault we want to detect, $x \in \mathbb{R}$ is the internal state and a is a known constant in the model, the initial state $x(0)$ is unknown.

a) Write the model using the transfer function, that is, find $G_u(p)$ and $G_f(p)$ in
$$y(t) = G_u(p)u(t) + G_f(p)f(t)$$

b) Write the model in the general form
$$H(p)x(t) + L(p)z(t) + F(p)f(t) = 0$$
that is, find the matrices $H(p)$, $L(p)$, and $F(p)$. The vector z of known signals is $z = (y, u)$.

Exercise 4.2.
Consider the same model as in Exercise 4.1.

a) The observation set
$$\mathcal{O} = \{z | \exists x \; H(p)x + L(p)z = 0\}$$
describes all observations from the process that could come from a system with no faults. It is good to know this set since when $z(t) \notin \mathcal{O}$ then we have detected a fault.

Describe the observation set $\mathcal{O}$ in implicit form, that is, provide the differential equation that the signals in z have to fulfill in order that z will contain in $\mathcal{O}$.

b) Describe the observation set $\mathcal{O}$ in explicit form, that is, provide the relationship in the time domain that $z(t)$ needs to fulfill in order to z will contain in $\mathcal{O}$. This means that the differential equation from (a) needs to be solved.

Hint: Solution to a first order differential equation

$$\dot{x} + ax - u = 0$$

is

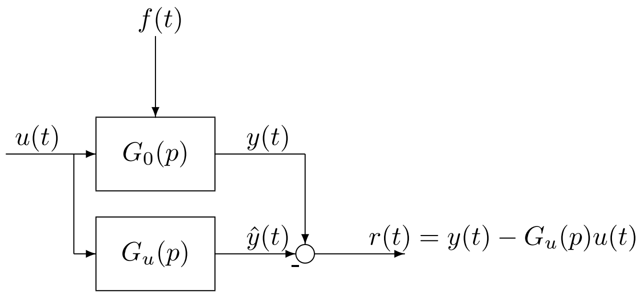
$$x(t) = x(0)e^{-at} + \int_0^t e^{-a(t-\tau)}u(\tau)d\tau$$

c) Construct a consistency relation for the system that can be used to test if $z \in \mathcal{O}$ or not.

d) Use the answer in (c) to construct a residual generator on state-space form where derivatives of z is not used in the calculation of the residual.

Exercise 4.3.
Consider, once again, the system in Exercise 4.1.

a) Assume that $a = 1$ and denote the true system transfer function as $G_0(p)$. Show with the help of the definition of a residual generators that if one generates a residual according to



and set the initial conditions in the residual generator to 0 then the filter is a residual generator.

b) Assume $a = -1$. Show that the expression in (a) is not an residual generator.

c) Derive a residual generator for the case $a = -1$.

Exercise 4.4.

Consider the static model

$$
\begin{aligned}
x_1 &= -3u + d \\
x_2 &= 2u - d \\
y_1 &= 3x_1 + f \\
y_2 &= x_1 + 2x_2
\end{aligned}
$$

where y_i and u are known signals, d is an unknown disturbance and x_i unknown internal variables. The signal f models a fault.

a) Consider the no fault case, that is $f = 0$. Find a consistency relation (and thereby a residual generator) that have decoupled the disturbance d and the unknown internal states x_i. This can be done by hand.

   Write the computational form of the residual generator.

b) Now, assume that $f \neq 0$, write the internal form of the residual generator.

(D) c) Write the model equations on the form

$$
M \begin{pmatrix} x_1 \\ x_2 \\ d \\ y_1 \\ y_2 \\ u \end{pmatrix} = 0_{4\times 1}
$$

   Perform Gaussian elimination on the matrix M, that is, transfer the matrix M to right triangular form by operations on rows. This can either be done by hand or by QR-factorization in Matlab.

   How can the results in (a) be seen in the right triangular matrix from of M?

d) In (c) the variables were arranged in such a way that x_1 was first and then x_2, d, y_1, y_2, u. Why was this sequence picked? Provide an other possible sequence of the variables, please comment the results.

Exercise 4.5.
The usage of the complex Laplace-variable s and the differential operator p are mixed in the next. The reason for this is that it formally is important to separate these two apart from each other. This exercise will illustrate why this is important.

a) Assume that the linear model

$$
y = \frac{b(p)}{a(p)} u \tag{4.1}
$$

   and the polynomial $a(s)$ and $b(s)$ share the same roots, that is, there exists a polynomial $q(s)$ that is contained in both $a(s)$ as well in $b(s)$. The transfer function can then be written as

$$
G(s) = \frac{b(s)}{a(s)} = \frac{b'(s)q(s)}{a'(s)q(s)} = \frac{b'(s)}{a'(s)} \tag{4.2}
$$

Explain why

$$y = \frac{b'(p)}{a'(p)} u$$

does not describe the same solution set as (4.1).

b) Define the solution set for (4.1) and (4.2)

$$\mathcal{O}_1 = \{(y(t), u(t)) | y(t) \text{ and } u(t) \text{ satisfies (4.1)}\}$$
$$\mathcal{O}_2 = \{(y(t), u(t)) | y(t) \text{ and } u(t) \text{ satisfies (4.2)}\}$$

Which subset relations are between the two sets $\mathcal{O}_1$ and $\mathcal{O}_2$?

Exercise 4.6.
Assume that we want to generate residuals based on the nominal model

$$y = \frac{1}{p+1} u$$

For linear systems there exists strong relations between constructions based on state-space-observers and constructions based on consistency relations. This exercise tries to clarify this relationship.

a) Derive a consistency relation and construct a first order residual generator based on consistency relation, that is, the residual generator shall be able to be written on state-space form with 1 state. Parameterize the residual generator in such a way that the pole in the residual generator is placed in $s = -\alpha$. Write the residual generator on either as a transfer function or on state-space form.

b) Write the model on state-space form.

c) Construct a first order residual generator based on state-space-observer for the system.

d) Clarify the relationship between the constructions in (a) and (c) by showing how the parameters in the solution for the observer case shall be chosen so they become identical to the solution based from the consistency relation.

e) In (d) it is straightforward to find an one-to-one relation between the two methods. Sketch, that is no calculations, on one or several cases where it is not as straightforward to find such a relation as in the previous case.

   Hint: Think about the ordinal number and how a decoupling can be achieved in the solution for the observer and also for the solution in the consistency relation case.

   What is the consequence of decoupling of signals, for example regarding fault isolation?

Exercise 4.7.

a) Write the single-input-single-output (SISO) model

$$2\dot{y} + y = \dot{u} + u$$

as a transfer function matrix using the differential operator p. Write also the model on state-space form.

b) Write the SIMO-model

$$\dot{y}_1 + y_1 = u$$
$$\dot{y}_2 + 2y_2 = u$$

on state-space form and as a transfer function matrix.

c) Introduce an additive fault in the sensor in (a) and write the whole model on state-space form.

d) Introduce additive faults in the actuators (f_1 och f_2) and one disturbance d in the first sensor in (b). Write the whole model on state-space form.

e) Find a residual generator that decouples the disturbance d in Exercise-(d). The dynamic of the residual generator shall be chosen that the residual can detect a fault with a time constant of about 0.3 seconds. Write the residual generator on state-space form.

Exercise 4.8.
Consider the SISO-system that is described by the following transfer function:

$$y = \frac{1}{p+1}u + f$$

where y is the measurement signal, u control signal, and f a model of a fault in the sensor.

a) Write the differential equation that corresponds to the transfer function.

b) Assume that not only y is known, but also $\dot{y}, \ddot{y}, y^{(3)}, \ldots$ and also u. Write, from (a), the computational form for a residual generator. This kind of relation is in the compendium called a *consistency relation*.

c) What happens to the residual if a fault occurrs, that is, write the internal form for the residual generator.

d) Disregard the assumptions that the derivatives of u and y are known. Find a new, implementable (for example on state-space form) residual generator. Write both the computational form and the internal form for the residual both in the time domain and in the frequency domain.

Exercise 4.9.
Assume that a consistency relation for a system has been constructed

$$\ddot{y}(t) + c_1\dot{y}(t) + c_2 y(t) - c_3\dot{u}(t) - c_4 u(t) = 0$$

To avoid to estimate derivatives it is wanted to generate the residual r according to the differential equation

$$a(p)r(t) = b_1(p)y(t) + b_2(p)u(t) \qquad (4.3)$$

where $b_1(p) = p^2 + c_1 p + c_2$, $b_2(p) = -c_3 p - c_4$, och $a(p)$ are polynomials that are design choices for the user.

a) Provide which demand is put on the polynomial $a(p)$ in order to r shall be a residual.

b) Realize (4.3) on state-space form. Use, for example, observable canonical form for the realization.

Exercise 4.10.
Assume that we have the same system as in Exercise 4.8, with the addition that we also measure the integral of y from Exercise 4.8. This can correspond to that y (from Exercise 4.8) is the angular velocity in a rotating machine and both the current angle, φ, and the angular velocity, ω, of the machine are measured. The new vector of measurement becomes then

$$y = \begin{pmatrix} \omega \\ \varphi \end{pmatrix}$$

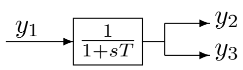
The model is then

$$y = \begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{bmatrix} \frac{1}{p+1} \\ \frac{1}{p(p+1)} \end{bmatrix} u$$

a) The upper part of the new system is the same system as in Exercise 4.8, thus, the residual generator from Exercise 4.8.-(d) can be used, without modifications, for this extended system. However, with the new sensor there is more possibilities, that is, more residual generators. Find, at least, one more residual generator compared to the ones found in Exercise 4.8.

b) How many linear independent residual generators can be found? Motivate your answer.

Exercise 4.11.
Assume a system according to the schematics below, where y_i are known measured signals



Describe the set of consistency relations that exist for the system. How large is the largest set of linearly independent consistency relations.

Find such a largest set and comment if there are any other consistency relations that could be of interest and why.

Exercise 4.12.
This task aims at illustrating the relation between decoupling and fault isolation. Consider a second order rotation systems where angle φ and angular velocity ω are measured, e.g:

$$\begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{bmatrix} \frac{1}{p+1} \\ \frac{1}{p(p+1)} \end{bmatrix} u$$

The model can be written in matrix form

$$\begin{bmatrix} -(p+1) & 0 \\ 1 & -p \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ u \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$

where the signals f_i model faults in the two sensors and w_i are state variables.

a) Assume that we want to generate a residual that isolates fault f_1 from fault f_2. Complete the residual structure below

	f_1	f_2
r		

with a residual that enables the wanted isolation property.

b) Rewrite the model on the following form

$$H(p)x + L(p)z + F(p)f = 0$$

so that a residual for the model is consistent with the structure in task a

Exercise 4.13.
Consider a second order system modeled as

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_f f \\ y &= Cx + D_u u + D_f f \end{aligned}$$

where u, f and y are vector valued signals. The dimension of both u and y is 2. Assume that only one fault can occur simultaneously and that one element in the vector f corresponds to one fault.

a) Assume that the column for f_1 in

$$\begin{bmatrix} B_f \\ D_f \end{bmatrix} \tag{4.4}$$

is linearly dependent of the column for another fault f_2. Can we isolate these faults from eachother?

b) Assume that we have three faults, f_1, f_2 och f_3, (i.e. the dimension of f is 3). Further, assume that none of the columns in (4.4) is parallel with another column but (4.4) despite this does not have full column rank ((4.4) has to have rank 2). Can we isolate these 3 faults from eachother?

c) What is the maximum dimension of f, i.e. the maximum number of faults, if we are to be able to isolate each and every one of the faults in the vector f?

Exercise 4.14.
Consider the following linear system with two sensors, one actuator and a modeled sensor fault in sensor 2.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{p+a} \\ \frac{1}{(p+b)(p+a)} \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f$$

The model is parametrized by two model parameters, a and b.

a) Find a residual generator that only uses the 2nd row in the model, i.e. a residual generator that only uses u and y_2 and disregards sensor y_1.

b) Assume that we do not know the exact values of the parameters a and b. Show that the uncertainty in both parameters will affect the performance of the residual.

c) Find a residual that is only affected by the uncertainty in one of the parameters, with the same internal form as the residual generator found in task a.

d) Which residual is better if the main model uncertainty is in the parameter a

Exercise 4.15.
Again consider the model in Task 4.14. A state-space formulation of the model is given by the following equations

$$\dot{w} = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -ab & -(a+b) \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

a) Write the model on the form

$$H(p)x + L(p)z + F(p)f = 0$$

where x are unknown signals, z known signals and f the faults we want to detect.

b) Compute, by hand, a basis $N_H(s)$ for the left null space to the matrix $H(s)$. Write the expression $N_H(p)L(p)z = 0$ and relate it to the solution to Task 4.14.

c) Redo the computions in Matlab to verify your own computations. Assign reasonable values to the parameters a and b. Comment on eventual differences compared to your hand calculations in task b.

Exercise 4.16.
Assume that we have the same model as in Task 4.10, but with modeled faults in both sensors.

$$y = \begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{bmatrix} \frac{1}{p+1} \\ \frac{1}{p(p+1)} \end{bmatrix} u + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

a) Find a basis for all residual generators that exist for the system.

b) Construct realizable residual generators based on the result from task a. Discuss the selection of the free parameters.

c) Use Matlab to compare the results with your answers in tasks a and b. Produce relevant plots. (D)

Exercise 4.17. (D)
Consider a model of an airplane with inputs and outputs according to

Inputs		Outputs	
u_1:	spoiler angle [tenth of a degree]	y_1:	relative altitude [m]
u_2:	forward acceleration [ms^{-2}]	y_2:	forward speed [ms^{-1}]
u_3:	elevator angle [degrees]	y_3:	Pitch angle [degrees]

The model has the state matrices:

$$A = \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -0.12 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.665 \\ 1.575 & 0 & -0.0732 \end{bmatrix}$$

$$C = [I_3 \; 0_{3\times 2}] \qquad D = 0_{3\times 3}$$

The aim of this task is to model faults in sensors and actuators, construct a residual generator that decouples 2 of these and finally evaluate the constructed residual generator.

a) Introduce model for sensor and actuator faults.

b) How does the state description look like when faults in sensor 3 and actuator 3 should be decoupled? Write B_f, D_f, B_d and D_d.

c) Form the matrices $H(p)$, $L(p)$ och $F(p)$.

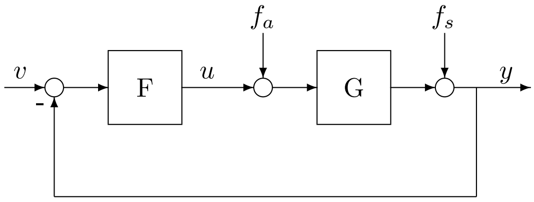
d) Find a polynomial basis $N_H(s)$ for the left nullspace to $H(s)$, using the command `null`.

e) Form a residual generator, i.e. select appropriate matrices γ and $d(s)$ that form the residual generator $R(s) = d^{-1}(s)\gamma N_H(s)L(s)$. How does$d(s)$ have to chosen in order to be realizable? Plot bode diagram for the residual generator $R(s)$ and the transfer function from fault to residual.

f) By selecting γ and $d(s)$ the residual generator can be shaped. Which properties are desirable and how can γ and $d(s)$ be used to accomplish this.

g) Why are faults in sensor 1 especially hard to detect? Can this be predicted?

Exercise 4.18.
The supervised system is often operated in closed loop, according to the schematics below:



where G represents the system and F the controller. It is therefore of interest to see how the controller affects the diagnosis problem. Assume that both the system and the controller are linear. Further assume that we have a model $\hat{G}(s)$ of the real system $G(s)$, i.e. $G(s) = \hat{G}(s) + \Delta G(s)$ where $\Delta G(s)$ is unknown. Let the residual r be $r = y - \hat{G}u$.

Decide whether the controller selection affects the residual in the case of:

a) $G(s) = \hat{G}(s)$, dvs. $\Delta G(s) = 0$ (no model errors)

b) $\Delta G(s) \neq 0$

c) Explain the results in a and b with words, and try to outline how the results affect the control design.

Exercise 4.19.
Assume a rotating system where both angle φ and angular velocity ω are measured, i.e.:

$$\begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{bmatrix} \frac{1}{p+1} \\ \frac{1}{p(p+1)} \end{bmatrix} u$$

Explain in simple manner why a additive fault in the angle sensor cannot be strongly detectable.

Exercise 4.20.
Consider a system described by the following transfer functions:

$$G_u(p) = \begin{bmatrix} \frac{2}{p+1} \\ \frac{1}{p+1} \end{bmatrix} \quad G_d(p) = \begin{bmatrix} \frac{1}{p+2} \\ \frac{1}{p+2} \end{bmatrix} \quad G_f(p) = \begin{bmatrix} \frac{2p+1}{p+2} \\ \frac{1}{p+2} \end{bmatrix}$$

a) Write the system in the standard form $H(p)x + L(p)z + F(p)f = 0$.

b) Use the detectability criterion from the compendium to decide if the fault is detectable. Note that this can be done with simple mental arithmetic.

c) Verify b) in MATLAB. (D)

d) Is the fault strongly detectable? Do the calculations by hand.

e) Verify d) in MATLAB. (D)

Exercise 4.21.
Show that there does not exist any asymptotically stable linear residual generator (i.e. all poles are strictly in the left half plane), where additive sensor faults are strongly detectable.

a) $y = \frac{p+1}{p(p+2)}u + f_s$

b) $y = \begin{bmatrix} \frac{1}{p} & \frac{1}{p+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + f_s$

c) Show tasks a and b through hand calculations, *without* using the detectability criterion from the compendium.

Hint: Use the fact that all residual generators can be written

$$r = \frac{A(p)y + B(p)u}{C(p)}$$

Exercise 4.22.

a) Assume a model of the following form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_d d \\ y &= Cx + D_u u \end{aligned}$$

Assume there exists matrices $P \neq 0$, A_z, $L_1 \neq 0$, $L_2 \neq 0$, K so that

$$PA = A_z P \quad \wedge \quad L_1 C = L_2 P \quad \wedge \quad (A_z - KL_2) \text{ stable} \quad \wedge \quad PB_d = 0$$

Show that the following residual generators decouple d

$$\begin{aligned}\dot{\hat{z}} &= A_z\hat{z} + PB_u u + K(L_1 y - L_1 D_u u - L_2\hat{z})\\ r &= -L_2\hat{z} + L_1 y - L_1 D_u u\end{aligned}$$

Hint: The residual generator is an observer for $z = Px$.

b) Construct a residual generator according to task a, for the model

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-1 & -1/2\\ 0 & -2\end{bmatrix} x + \begin{pmatrix}1\\1\end{pmatrix} u + \begin{pmatrix}1\\2\end{pmatrix} d + \begin{pmatrix}0\\1\end{pmatrix} f\\ y &= x\end{aligned}$$

c) Change the A-matrix from taks a to

$$A = \begin{bmatrix}-1 & -1/3\\ 0 & -2\end{bmatrix}$$

Show that there does not exist a solution according to task a. Also, find a residual generator for the system by hand.

Exercise 4.23.
This task is a theoretical task to aid in the solution of Task 4.24.

Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices where B is positive definite. Consider the problem of finding a scalar λ_i and the vector z_i such that

$$Az_i = \lambda_i B z_i$$

This is the *symmetric-definite generalized eigenvalue problem*[1]. The task is to show that the eigenvectors z_i can always be scaled so that:

$$z_i^T B z_j = \delta_{ij} = \begin{cases}1 & i = j\\ 0 & i \neq j\end{cases}$$

samt

$$z_i^T A z_j = \lambda_j \delta_{ij}$$

Hint: The result is general, however, to make the task easier it is ok to assume that $\lambda_i \neq \lambda_j$ for $i \neq j$ and that all eigenvalues are nonzero.

Exercise 4.24.
The task is about making an optimal trade-off between fault sensitivity and attenuation of disturbances in the case where perfect decoupling of disturbances is not possible. This can be for instance the case when the process is equipped with too few sensors.

The Chow-Willsky algorithm is described in the course compendium and the equation (6.31) describes the dependence of the consistency relation w.r.t. the different included variables. If perfect decoupling were possible, w would be

[1]Compare with the normal eigenvalue problem, received when $B = I$ and the symmetry constraint in A and b is removed.

selected so that (6.32) is fulfilled and that the x and V terms would disappear, so that only the influence of the faults remained. Assume that (6.32) is not feasible, i.e. perfect decoupling of both x and d is not possible. Then a consistency relation can be created where the influence of the disturbances V is as small as possible, while making the sensitivity to faults as large as possible.

This can be achieved by first decoupling all internal states x and then using the available freedom to do the trade-off between sensitivity to faults and attenuation of disturbances,

a) Show that decoupling of x is always possible and parametrize all w such that x is decoupled in (6.31). I.e. decide the matrix N in the expression

$$w^T = \zeta^T N$$

such that for all w such that x is decoupled, there exists a corresponding ζ.

b) The internal form for a consistency relation with a w that decouples x then becomes:

$$h(y,u) = \zeta^T NHV + \zeta^T NPF$$

Explain how the optimization problem below, does a trade-off between sensitivity in the faults and attenuation of the disturbances in the residual.

$$\max_{\zeta} \frac{\|\zeta^T NP\|_2^2}{\|\zeta^T NH\|_2^2}$$

c) Solve task 4.23 (or simply use the result) to find a method to solve the optimization problem in task b.

Hint:

1. The euclidean norm can be written as a matrix multiplication. For instance $\|x^T A\|_2^2 = x^T A A^T x$
2. Rewrite the optimization criterion on the form

$$\max_{\zeta} \frac{\zeta^T A \zeta}{\zeta^T B \zeta}$$

and switch basis $\zeta = Zv$. To find the transformation matrix Z, use the results from task 4.23.

d) In the file `422.mat` there are state matrices for a model

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_d d + B_f f \\ y &= Cx + D_u u + D_d d + D_f f \end{aligned}$$

where perfect decoupling is not possible. Use Matlab to do a design with optimal approximate decoupling of the disturbances d. (D)

Chapter 5

Nonlinear Residual Generation

Exercise 5.1.
Assume a nonlinear first order system in state-space form

$$\begin{aligned}\dot{x} &= -xu \\ y &= x^2\end{aligned}$$

where y and u are known signals and x is an unknown internal state.

Find a consistency relation for the system.

Exercise 5.2.
Assume a nonlinear first order system, in the fault free case described on state-space form according to

$$\begin{aligned}\dot{x} &= -xu \\ y &= x\end{aligned}$$

where y and u are known signals and x is an unknown internal state.

a) Construct a residual generator using a state observer that detects faults for the system

   Choose the observer gain so that stability of the observer is guaranteed in the fault free case.

b) Assume an additive fault in the sensor, i.e. the sensor equation is described by

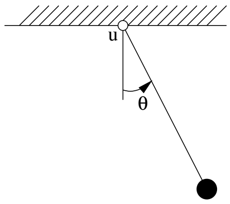
$$y = x + f$$

   State the internal form of the residual generator when K is chosen as in task a.

c) Assume a situation with a constant fault $f = f_0$ and a constant control $u = u_0$. Explain, using the answer in task b, what/which u_0 that are problematic. Can this be seen in the model equation?

Exercise 5.3.
Observers are an important tool to generate residuals for nonlinear systems. A fundamental step in observer design is to ensure stability, i.e. that the by the observer estimated state converges towards the actual state.

For this aim, consider a simple pendulum with viscous damping in the joint, according to



Assume that the pendulum can be controlled by a torque at the joint. If the physical constants are set so the coefficients of the model are simple, the pendulum can be described by the differential equation

$$\ddot{\theta} + \dot{\theta} + \sin\theta = u$$

Assume the process is equipped with an angle sensor. In state-space form, with $x_1 = \theta$ and $x_2 = \dot{\theta}$, the model becomes

$$\begin{aligned} \dot{x} &= f(x,u) = \begin{pmatrix} x_2 \\ -x_2 - \sin x_1 + u \end{pmatrix} \\ y &= x_1 \end{aligned}$$

Construct a state observer of the form

$$\dot{\hat{x}} = f(\hat{x}, u) + K(y - \hat{x}_1)$$

and select the gain K so that the observer is stable.

There are several ways to prove stability, but a basic method is called Lyapunov theory. In short, the principle can be described as defining a measure of magnitude, a Lyapunov function, on the estimation error och show that the magnitude is decreasing with time.

One example of a Lyapunov functions that can be used in this case is $V(e_1, e_2) = e_1^2 + \beta e_2^2$ where $e_i = x_i - \hat{x}_i$ and $\beta > 0$. If one can select K so that $\dot{V} < 0$ one knows that $e(t) \to 0$ when $t \to \infty$.

Hint: The inequality written below might be of help.

$$0 \leq \frac{\sin x - \sin y}{x - y} \leq 1 \qquad -\pi/2 \leq x, y \leq \pi/2$$

Exercise 5.4.
Assume that the following two equations represent consistency relations for

two systems.

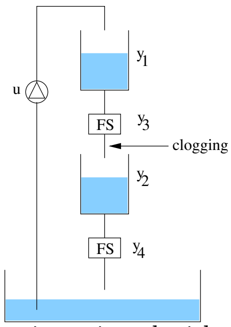
$$\ddot{y} - 2\dot{y} + 3\dot{u} - u = 0, \qquad \ddot{y}\dot{y} - yu = 0$$

where y and u are known signals while their derivatives are unknown.

In the linear case it is easy to use the relation to generate residuals, if dynamics is added according to Example 2.1 in the compendium.

Why isn't it as easy in the nonlinear case?

Exercise 5.5.
Consider a system of water tanks according to the figure below.



$$\begin{aligned}
\dot{h}_1 &= a_1 u - a_2\sqrt{h_1} \\
\dot{h}_2 &= a_2\sqrt{h_1} - a_3\sqrt{h_2} \\
y_1 &= h_1 \\
y_2 &= h_2 \\
y_3 &= a_4\sqrt{h_1} \\
y_4 &= a_5\sqrt{h_2}
\end{aligned}$$

The process is equipped with 4 sensors, y_1 and y_2 measure the level in the upper and lower tanks respectively. The sensors, y_3 and y_4 measure the flow out of the two tanks and are marked with FS (Flow Sensor) in the figure. Also given is a simple nonlinear model of the process where the state variables h_i represent the water levels in the two tanks and u is the control signal for the pump, i.e. the pump power

a) Model in an arbitrary manner, faults in the four sensors and in the pump.

b) Model clogging in the pipe between the upper flow sensor and the lower tank.

c) In chapter 6 consistency relations for linear systems where developed, i.e., relations of the type

$$g(y, \dot{y}, \ddot{y}, \ldots, u, \dot{u}, \ddot{u}, \ldots, f, \dot{f}, \ddot{f}, \ldots) = 0$$

Develop, by hand, such relations based on the model equations for the water tanks.

d) Assume that, not only y, but also $\dot{y}, \ddot{y}, \ldots$ (and the corresponding for u) are known. Write down residual generators based on the consistency relations developed in task c. Write down both the calculation form as well as the internal form of the residual generators.

Exercise 5.6.
Again consider the watertank system in Task 5.5 and assume that the derivatives of the known signals are so noisy that derivatives are hard to estimate. Denote the sensor faults $f_1, \dots, f_4$ and pump fault f_u.

a) Take a consistency relation from Task 5.5 that is sensitive to f_1 and f_u. Add dynamics and write the residual generator on state-space form where no derivatives of known signals are required in the calculation form.

b) Find a consistency relation that is sensitive to f_3 and f_u and no other faults. Use the same procedure as in task a to get rid of the derivative of y_3, illustrate why task b is more complicated than task a. The residual generator should work in all operating points, even when the upper tank is empty, i.e. when $h_1 = 0$.

Exercise 5.7.
Again consider the watertank system in Task 5.5 where residual generators were constructed by first deriving consistency relations and then in Task 5.6 handling that derivatives of known signals are unknown.

Now instead assume that we want to construct residual generators directly, using observer methodology. Denote the sensor faults $f_1, \dots, f_4$ and pump fault f_u. Construct, if possible, a residual generator that is

a) sensitive to f_1 and f_2 but not sensitive to f_u, f_3 and f_4.

b) sensitive to f_3 and f_2 but not sensitive to f_u, f_1 and f_4.

c) sensitive to f_2 but not sensitive o f_u, f_1, f_3 and f_4.

d) sensitive to f_2 but not sensitive to f_u, f_1, f_3 and f_4 where f_u is assumed to be constant/slowly varying.

e) Reflect over the differences and difficulties in designing residual generators using consistency relations versus observers for this system.

Exercise 5.8.
Consider the system

$$\begin{aligned}
\dot{x}_1 &= g_1(x_1, f_1, u) \\
\dot{x}_2 &= g_2(x_1, x_2) \\
y_1 &= h_1(x_1) \\
y_2 &= h_2(x_2) + f_2
\end{aligned}$$

where $x_1 \in \mathbb{R}^3$, $x_2 \in \mathbb{R}$, f_1 och f_2 are two modeled faults, g_i och h_i are nonlinear functions. The fault free case corresponds to $f_1 = f_2 = 0$.

a) Assume that both faults vary slowly and can be assumed constant. Construct an observer that estimates both faults.

b) Using the same constant fault assumption construct, through an observer, a residual generator where the residual is sensitive to fault f_2 but not f_1.

c) Redo task b but with sensitivity to f_1 and not f_2.

d) Which constant fault assumption, f_1 or f_2, can most easily be removed and still solve task b and c?

Exercise 5.9.
Consider a system described by the following differential equation

$$\dot{y} + \theta y - u = 0$$

where θ is a constant, i.e. $\dot{\theta} = 0$.

Find a consistency relation that is independent of θ. That is, find a function g such that the following holds

$$\forall \theta. \quad g(y, \dot{y}, \ddot{y}, \dots, y, \dot{u}, \ddot{u}, \dots) = 0$$

Reflect on how the task relates to diagnosis.

Hint: Differentiate!

Exercise 5.10.
Consider the following nonlinear differential equation

$$\begin{aligned} \dot{x} &= -x^2 + u \\ y &= x^3 \end{aligned}$$

a) Validate that a solution to the model satisfies the consistency relation

$$\dot{y}^3 + 27\dot{y}y^2u + 27y^4 - 27y^2u^3 = 0$$

$\star$b) Derive the above consistency relation in the manner outlined in the compendium, that is, eliminate the state variable x from the equations

$$\begin{aligned} y - x^3 &= 0 \\ \dot{y} - 3x^2\dot{x} &= \dot{y} - 3x^2(-x^2 + u) = 0 \end{aligned}$$

$\star$c) Represent the model equations with the left hand side of the equations in task b, i.e.

$$f_1 = y - x^3 \qquad f_2 = \dot{y} - 3x^2(-x^2 + u)$$

The elimination stage in task b can be written as a linear combination of f_i, i.e. there are polynomials h_1 and h_2 such that

$$\dot{y}^3 + 27\dot{y}y^2u + 27y^4 - 27y^2u^3 = h_1 f_1 + h_2 f_2$$

Since $f_i = 0$ it will also hold that $h_1 f_1 + h_2 f_2 = 0$, i.e. the consitency relation is derived.

Find the polynomials h_1 and h_2.

Exercise 5.11.
Assume the static system

$$y(t) = \theta_1 u_1(t) + \theta_2 u_2(t)$$

where θ_i are constants modeling two different faults. In the fault free case $\theta_i = 1, i = 1, 2$.

A residual should be constructed where variations in θ_2 are decoupled. For simplicity we assume that u_i and y are measured on two different occations (generalization to N points is possible).

There are two obvious ways to construct r

1. Assume $\theta_1 = 1$ and estimateθ_2 (e.g. using least squares) from the two measurements. Then insert $\hat{\theta}_2$ in the equations to get a residual generator that does not react when $\theta_2 \neq 0$.
2. Write the model equations for both time steps, i.e.

$$\begin{aligned} y(t) &= \theta_1 u_1(t) + \theta_2 u_2(t) \\ y(t-1) &= \theta_1 u_1(t-1) + \theta_2 u_2(t-1) \end{aligned}$$

   Derive a residual generator by elimination θ_2 from the equation system.

a) Write a residual generator according to principle 1

b) Write a residual generator according to principle 2

c) Find the relations between the residual generators in task a and b.

Exercise 5.12.
Here the air intake system in a regular SAAB production engine for passenger cars should be supervised. The system is described by nonlinear differential equations and the model is described in detail further down in the task.

The task is to construct a nonlinear diagnosis system, capable of detecting faults in one or more of the sensors in the system.

- The throttle angle sensor
- The air massflow sensor
- The pressure sensor in the intake manifold

To test the diagnosis system there are fault free measurements of the actual engine available in the file `faultfree.mat`. Measurements with faults are in the file `fault.mat`. The measurement with faults is 60 s long and fault free the first 10 s, then the three faults occur, one by one, for 15 s each. Between each of the faults there is a 2 s fault free pause. A Simulink model of the air intake system is in the file `nonlinsim.m`. Note that the actual throttle angle α is unknown (that is, the angle given by the accelerator pedal), only the (possibly incorrect) sensor signal α_s is known.

All files, including Simulink files and measurement data, are in the library `Engine/`. The functions f and g in the model are in the files `fmat2.m` and `gmac2.m`, and the constant in the model

$$\frac{RT_{man}}{V_{man}} \approx 3.5964$$

A model for the air intake system of a SI-engine

The SI(Spark Ignited)-engine is a non-linear plant and for the purpose of diagnosis, a simple and accurate model is desirable. In the air system application there is no need for extremely fast fault detection, therefore a so called *mean value model* is chosen. This means that no within cycle variations are covered by the model. The engine is a 2.3 liter 4 cylinder SAAB production engine. The measured variables are the same as the ones used for engine control. A schematic picture of the whole engine is shown in Figure 5.1. The engine has electronic throttle control (drive-by-wire), which is basically a DC-servo controlled by a PID controller. The part that is considered to be the air intake system is everything to the left of the dashed line. Also the engine speed must be taken into account because it affects the amount of air that is drawn into the engine.

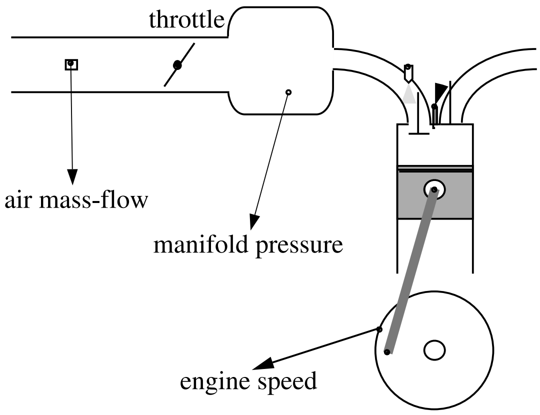


Figure 5.1: The basic SI-engine.

The model of the air intake system is continuous. It is derived from the ideal gas law and has one state which is the manifold pressure. The process inputs are the throttle angle α, and the engine speed n. The outputs are throttle angle sensor α_s, mass air flow sensor $\dot{m}_{air,s}$, and manifold pressure sensor $p_{man,s}$. The equations describing the fault free model can be written as

$$\dot{p}_{man} = \frac{RT_{man}}{V_{man}}(\dot{m}_{air} - \dot{m}_{ac}) \tag{5.1}$$

$$\dot{m}_{air} = f(p_{man}, \alpha) \tag{5.2}$$

$$\dot{m}_{ac} = g(p_{man}, n) \tag{5.3}$$

The variables and its units are summarized and explained in Table 5.1. The model consists of a physical part, equation (5.1), and a black box part, the functions (5.2) and (5.3). Even if variations in ambient pressure and temperature do affect the system, they are here assumed to be constant.

p_{man} [kPa]	manifold pressure
R [J/(g K)]	the gas constant
T [K]	manifold air temperature which is assumed to be equal to the ambient temperature
V [m^3]	manifold volume
$\dot{m}_{air}$ [kg/s]	air mass flow into the manifold and is equal to the air flow past the air mass flow meter
$\dot{m}_{ac}$ [kg/s]	air mass flow out from the manifold
f	static function describing the flow past the throttle
g	static function describing the flow into the cylinders
α [deg]	throttle angle (unknown)
n [rpm]	engine speed

Table 5.1: Symbols and units.

Chapter 6
Multiple fault isolation

Exercise 6.1.
The purpose of this exercise is to give understanding of minimal diagnoses. Which of the diagnoses in Table 1.1 in the course literature are minimal diagnoses?

Exercise 6.2.
This exercise explains the basic ideas of diagnosis and conflict. The Polybox example is shown in Figure 6.1. Assume that we observe that $a = 3$, $b = 3$,

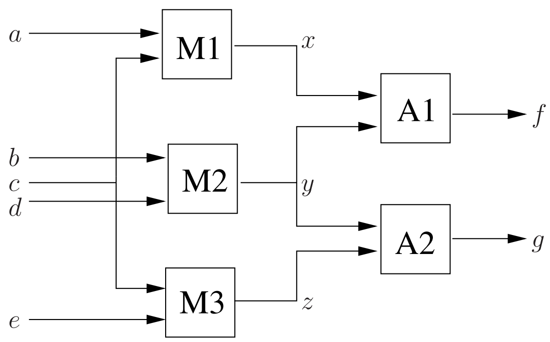


Figure 6.1: Polyboxexemplet.

$c = 2$, $d = 1$, $e = 5$, $f = 9$ and $g = 12$.

a) Verify using Definition 2.1 in the course literature that

$$OK(M1) \wedge OK(M2) \wedge OK(M3) \wedge OK(A1) \wedge \neg OK(A2)$$

is a diagnosis. Write down explicitly the three sets in the definition, i.e., the sets describing the model $\mathcal{M}$, the observations *obs*, and assigned modes $\mathcal{D}$.

b) Derive a conflict using Definition 3.1 in the course literature.

c) Use definition 3.2 to evaluate if the derived conflict is minimal.

Exercise 6.3.
This exercise gives understanding how diagnoses can be computed given a set of conflicts. Consider a system with three components, A, B, and C, and the following conflicts

$$\pi_1 = OK(A) \wedge OK(B)$$
$$\pi_2 = OK(B) \wedge OK(C)$$

are detected, i.e., residuals sensitivite to faults in A and B, and B and C, respectively, have triggered. Also, assume that each component can either be intact OK or broken $\neg OK$ and there are no models for broken components.

a) Use Theorem 3.3 to show that

$$\mathcal{D}_1 = OK(A) \wedge \neg OK(B) \wedge OK(C)$$
$$\mathcal{D}_2 = \neg OK(A) \wedge OK(B) \wedge \neg OK(C)$$

are diagnoses but

$$\mathcal{D}_3 = OK(A) \wedge OK(B) \wedge \neg OK(C)$$

are not.

b) Instead of using the logic notation in task a, use the set notation and Theorem 3.5 to show tha same thing as in task a.

c) Compute the single-fault diagnoses by taking the intersection of the negated conflicts.

Exercise 6.4.
This exercise show how the information about the fault sensitivities of the residuals in a decision structure can be used for multiple-fault isolation by first computing conflicts and then diagnoses.

a) Given the following decision structure

	A	B	C
r_1	0	X	X
r_2	X	0	X
r_3	0	X	0

write, using logic notation, generated conflicts when no residual have triggered and when the residual r_2 has triggered, respectively.

b) Assume that all three residuals have triggered. Compute the diagnoses by using the decision structure as before, i.e., by computing the intersections of the decisions from each test. Also, compute the minimal diagnoses by first writing down the conflicts and the use Theorem 3.3. Compare and comment the differences between the diagnoses computed from each method.

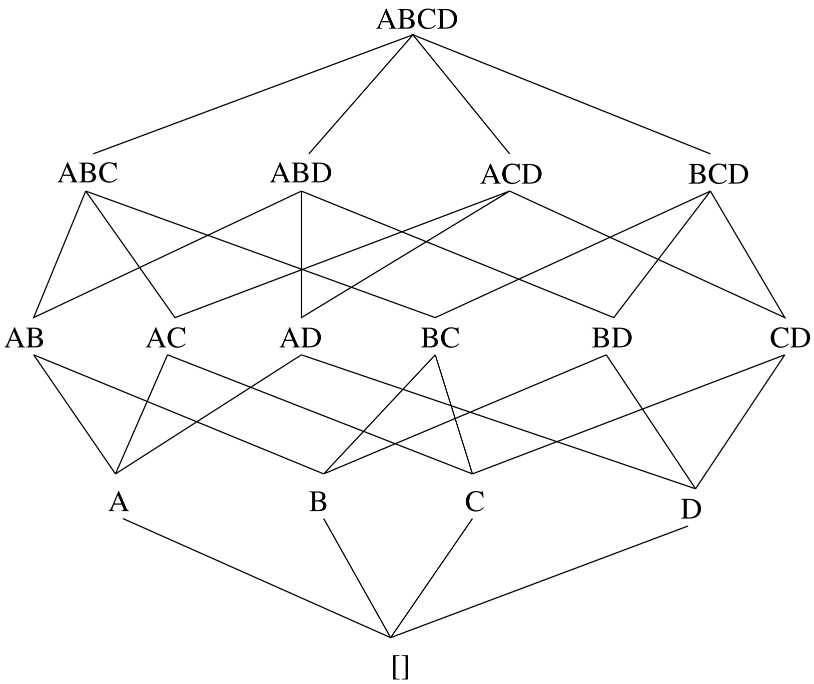
Exercise 6.5.
This exercise deals with multiple-fault isolation.Consider a system with components A, B, C, and D. Each component has two modes, OK or $\neg OK$. The behavior is unknown in mode $\neg OK$.

a) Assume that we have a diagnosis system based on a set of tests and the decision structure for 5 of the tests is:

	A	B	C	D
T_1	X	X		
T_2			X	X
T_3	X			X
T_4	X		X	
T_5		X		X

Write doen for each test which conflict is generated when the test quatity exceeds its threshold.

b) Assume that the tests triggers in ascending order, i.e., $T_1, T_2, \ldots, T_5$. Use the algorithm in Chapter 3 in the course literature to compute the minimal diagnoses. You can use the lattice to simplify your calculations.



Exercise 6.6.
This exercise illustrates the comleity problems that occurs when multiple-fault isolation is performed by extending the decision structure with all multiple faults and computing diagnoses using intersections according to (3.17). A system has 30 components and each component has four behavior modes, including NF.

a) The number of columns in the extended decisions structure is equal the total number of system behavior modes. How large is the total number of system behavior modes?

b) The intersection operation in (3.17) contains sets S_i describing the decisions taken from each test. Assume that diagnosis test 1 test the null hypothesis that component 1 is in mode NF. How many elements would the set S_1^1 contain?

c) If only, fault-free, single fault, and double fault modes are considered, what will the total number of system behavior modes be?

d) With the constraint in task c, how many elements are in set S_1^1?

Exercise 6.7.

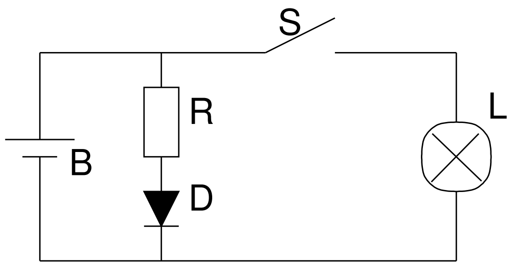


Figure 6.2: A simple circuit.

The circuit in Figure 6.2 contains 5 components: a switch (S), a resistor (R), a LED (D), a battery (B), and a lamp (L).

The resistor and the LED are assumed to work and the other components can have the following modes: The switch can be OK $OK(S)$, stuck open $SO(S)$, or stuck closed $SC(S)$. The battery can be $OK(B)$ or discharged $\neg OK(B)$. The lamp can be OK $OK(L)$ or broken $\neg OK(L)$. Also multiple-faults can occur. Observations are the requested position of the switch, if the LED is lit, and if the lamp is lit.

a) Write down all minimal conflicts for all 8 combinations of the three observations.

b) For all combinations of observations, write down all diagnoses. Indicate for each observation, which diagnoses are minimal.

c) Are all minimal conflicts characterized by the minimal conflicts?

d) Are all diagnoses characterized by the minimal diagnoses?

Exercise 6.8.
Two lamps, L_1 and L_2 are connected in parallel over a battery B, see Figure 6.3. Assume that fault models are missing, i.e., the behavior of the components are only known in the fault-free case. If the battery is working, it generates a

voltage. The voltage is denoted E such that $E =$ true if there is a current and false otherwise. Assume that only lamp L_2 emits light.

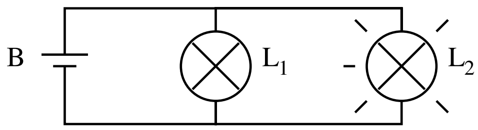


Figure 6.3: A simple circuit with two lamps and a battery.

a) Write the model and the observations as logical expressions.

b) Use Definition 2.1 to compute the diagnoses and Definition 2.2 to compute the minimal diagnoses?

c) What happens if we assume "exoneration"? Describe with logics what is included to the model. Use the expressions to compute new diagnoses.

d) What must be added to the model to get an intuitively correct diagnosis?

e) Improve the model according to your answer to task d. Again, use the definition of a diagnosis to compute the diagnoses.

Exercise 6.9.
Two inverters are connected in series according to Figure 6.4. Each inverter has three behavior modes: the inverter is OK OK, the output is grounded $SA0$ and the inverter is short-circuited $SHORT$. Assume that x can be selected freely and the system behavior mode cannot change.

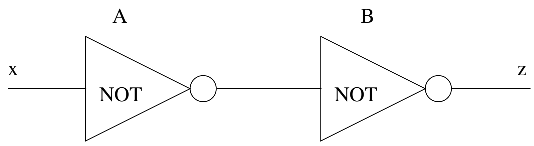


Figure 6.4: Two inverters connected in series.

a) First, let the input be $x = 1$ and the output $z = 0$. Find all conflicts and compute all diagnoses.

b) To reduce the number of diagnoses in a, the input is set to $x = 0$. The value of z becomes 1. Find all new conflicts and write down the diagnoses.

Exercise 6.10.
Prove Theorem 3.2 and Theorem 3.3.

Exercise 6.11.
Prove Theorem 3.5 and Theorem 3.6.

Exercise 6.12.
A system has three different faults. Assume that we have three test quantities, sensitive to each fault.

a) Write down the decision structure given that only single-faults can occur and noise affects the decisions. The behavior modes are denoted NF, F_1, F_2 och F_3.

b) Assume now that also double-faults can occur, i.e., the behavior modes F_{12}, F_{23} och F_{13} are added. Extend the decision structure with these modes.

c) Which behavior modes can be isolated uniquely?

d) Propose new test quantities that would be needed to isolate all behavior modes.

Exercise 6.13.
Consider the polybox-example in Section 3.4.6. Assume that we work in a noise-free environment, for example, in a computer.

a) Show that $A1$ cannot be isolated from $A1\&M1$.

b) Show that there are fault magnitudes in mode $A1\&M1$ that are isolable from all fault magnitudes in mode $A1$.

c) Which faults in mode $A2\&M1$ can be explained by a fault in mode $M2\&M3$ and which faults in mode $M2\&M3$ can be explained by a fault in mode $A2\&M1$?

d) Consider the following four test quantities:

$$\begin{aligned} T_0 =& |f - ac - bd| + |g - bd - ce| \\ T_1 =& |g - bd - ce| \\ T_2 =& |f - g - ac + ce| \\ T_3 =& |f - ac - bd| \end{aligned}$$

Write down the decision structure for single and multiple-faults. Hint: realize that several modes are “identical” and can be considered in common, then the analysis will be much shorter.

e) Redo the tasks a-c by only using the decision structure from the answer in d.

f) Given the response $T_0 \neq 0$, $T_1 \neq 0$, $T_2 \neq 0$ och $T_3 = 0$, which are the correct diagnoses?

g) Assume that all X in the decisionstructure wrongly are changed to 1. Which diagnoses do you get for the same response as in f?

Chapter 7

Probabilistic Diagnosis

Exercise 7.1.

a) During an experiment, 20 medical therapists performing HIV tests were asked: what is the probability of HIV infection if a random person, not belonging to any risk group, leaves a positive HIV test?

In a population without risk behavior it is assumed that 1 of 10 000 is infected statistically. The test will always give a positive response if a person is infected. For a non-infected person, the probability is 1 in 10 000 to get a positive reponse.

What is the correct answer to the question? If the test is positive, how big is the change that the person really is HIV positive?

b) Assume that the person is from a risk group where 1 of 100 are infected. What will the answer be in task a?

c) Discuss how the previous tasks relate to the diagnosis problem.

Exercise 7.2. (D)
Consider a residual generator and residual r is constructed to detect a fault f_1 and an alarm is triggered when r exceeds a given threshold J.

First, assume that the fault f_1 is the only fault that can occur and the a priori probability of the fault $P(f_1) = 0.1$. The threshold J is selected such that the probability of false alarm given r is $P(\text{alarm}|FM = NF) = 0.05$ and the probability of detecting the fault is $P(\text{alarm}|FM = f_1) = 0.9$.

a) Write down the probabilities for $P(f_1)$ and $P(\text{alarm}|f_1)$.

b) It is observed that the test has triggered, i.e., $r > J$. Calculate the probability of a fault f_1.

It turns out after further analysis that another fault f_2 can occur, which can be assumed as commonly occuring as f_1, independent of f_1, and can trigger r. The sensitivity of the residual given single-fault and double-faults is determined as

$$P(\text{alarm}|FM = f_2) = 0.60$$
$$P(\text{alarm}|FM = f_1\&f_2) = 0.95.$$

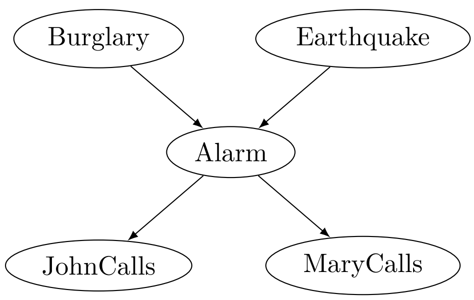
c) Write the conditional probability tables for $P(\text{alarm}|f_1, f_2)$ and $P(f_2)$. Again, assume that r triggers. Calculate the probabilities for each fault f_1 and f_2.

d) What is the probability that the test triggers?

e) Calculate the probability for each fault mode $FM \in \{NF, f_1, f_2, f_1\&f_2\}$ given that r has triggered.

f) Implement the problem using the software GeNIe and verify the results from previous tasks b, c, and d.

Exercise 7.3. (D)
The purpose of this task is to understand how conditional probability tables can be used to model more general situations and to use dedicated software to solve these type of problems. In one example, taken from (S. Russel and P. Norvig, 2003), there is a burglary alarm which also triggers to smaller earthquakes. If the alarm sounds, your neighbours, John and Mary, have promised to call you. This situation has five binary stochastic variables, B (Burglary), E (Earthquake), A (Alarm), J (John calls), and M (Mary calls). A Baeysian Network of this situation is given by



where the a priori probabilities of burglary and earthquake are 0.001 and 0.002 respectively and the other probability tables are

B	E	$P(A\|B,E)$
false	false	0.001
false	true	0.29
true	false	0.94
true	true	0.95

A	$P(J\|A)$
false	0.05
true	0.90

A	$P(M\|A)$
false	0.01
true	0.70

Use the software GeNIe to calculate the probabilities of

- burglary given that both Mary and John calls,
- earthquake given that both Mary and John calls,
- that Mary calls given a burglary,

- that John calls given no burglary,

i.e., the following probabilities

$$P(B|j),\ P(B|m),\ P(E|j,m),\ P(M|b),\ P(J|\neg b).$$

Exercise 7.4. (D)
Consider three test quantities T_1, T_2, T_3 to monitor three faults f_1, f_2, and f_3. The test quantities have the decision structure

	NF	f_1	f_2	f_3
T_1	0	0	X	X
T_2	0	X	0	X
T_3	0	X	X	0

From a large amount of measurements, probabilites that each test will trigger for each fault mode have been estimated, both single-faults and double-faults, which is summarized in the following matrix

	f_1	f_2	f_3	$f_1\&f_2$	$f_1\&f_3$	$f_2\&f_3$
T_1	$\underline{0.05}$	0.80	0.50	-	-	0.85
T_2	0.40	$\underline{0.05}$	0.85	-	0.90	-
T_3	0.90	0.65	$\underline{0.05}$	0.98	-	-

The underlined probabilities in the diagonal of the single-faults corresponds to the false alarm rates of each test quantity. If a fault does not affect a test quantity is represented by a dash. It means for example that

$$P(T_1|f_1\&f_2) = P(T_1|f_2)$$

The a priori probability of each fault is $P(f_i) = 0.1$ and the faults are assumed independent of each other.

a) Calculate the minimal diagnoses given the decision structure if T_1 and T_2 have triggered.

b) Implement the problem above using the software GeNIe. Use the stochastic variables f_1, f_2, and f_3, for each fault, and T_1, T_2, and T_3, for each test.

c) Assume that it is observed that T_1 and T_2 have triggered but not T_3. What is the probability of each fault? It is assumed that the faults occur independently of each other, explain why the probability of a fault-free system

$$P(NF|t_1, t_2, \neg t_3) = P(\neg f_1, \neg f_2, \neg f_3|t_1, t_2, \neg t_3)$$

and *cannot* be calculated as

$$\prod_i P(\neg f_i|t_1, t_2, \neg t_3).$$

d) The consistency-based diagnosis $f_1\&f_2$ has low probability, actually lower than, for example, $f_2\&f_3$. Explain why?

e) Assume that, similar to task c, T_1 and T_2 have triggered but T_3 has not been evaluated, i.e., we do not know if it has triggered or not. How will this affect the situation compared to c? Explain the difference.

Exercise 7.5. (D)
In task 7.4, the faults are modeled as binary stochastic variables. One alternative could be to model all *fault modes*, but it would scale badly since the number of fault modes grows exponentially with number of considered multiple-faults.

Still, it can be interesting to add variables to represent fault modes as illustrated in this task.

a) In task 7.4-c it was concluded that probability of a fault-free system cannot be directly calculated from the conditional probabilities of each fault. To automatically compute the probability of a fault-free system, add a new stocahstic variable, called Fault, in the Bayesian Network. The variable should be false if the system is fault-free and true if at least one fault is present. Fault will be a deterministic function of the fault variables. Write down the function and the probability of a fault-free system.

b) Assume that we want to use the Bayesian Network to isolate faults only, i.e., we know that there is a fault and want to compute the probabilities

$$P(F_i|t_1, t_2, \neg t_3, \text{fault}), \quad i = 1, 2, 3$$

Use the network from task a to calculate these probabilities. Comment what are the differences with respect to task 7.4-c.

c) In the same way, add noted for all 8 fault modes (fault-free, 3 single-faults, 3 double-faults, and triple-faults). Compute the probabilities given the test results.

Exercise 7.6. (D)
Model the system from task 7.4 using canonical models, *leaky-or*, according to the discussion in Section 8.3.1 in the course book and implement in the software GeNIe where the node type is called *NoisyMax* which is a generalized *Leaky/Noisy-Or*. Discuss the differences in the results, for example compared to b and c in task 7.4.

Exercise 7.7. (D)
A probability-based diagnosis system is implemented for a simple watertank system where an actuator controls a pump that pumps water in a tank and sensors measure the waterlevel and flow out of the tank. The purpose in this task is to practice probability-based modeling.

For simplicity, assume that the state of the flows and water level is either high or low. If the actuator is high, the probability is high that the water inflow is high and vice versa. A higher inflow in the tank increases probability of a high water level in the tank and a high water level a large outflow. The sensor values contain uncertainties.

a) Model the watertank and sensors as a Bayesian Network and implement it in a software. Verify that the model give reasonable results for different actuator values and observations. Suggested stochastic variables are: pump reference signal R, pump outflow U, tank water level H, tank outflow W, and sensors Y_h and Y_W.

Faults that should be detected are leakage in the tank, clogging in the outflow pipe, pump failure, and sensor faults. A leakage increase the probability of low water level and clogging that the outflow is low even though the water level is high.

b) Extend the model from the previous task with the different faults. Verify that the model gives reasonable results for different actuator values and observations.

Chapter 8

Fault Effect and Fault Tree Analysis

Exercise 8.1.

a) Gör en process-FMEA som beskriver en lagning av en cykelpunktering. De processsteg som åtminstone ska ingå är: sökning efter läckage, lagning av slang, kontroll av insidan på däcket och montering av däck på fälg.

Processsteg	Felsätt	Felorsak	Feleffekt	Riskanalys			
				OCC	SEV	DET	RPN

b) För att analysera resultatet från FMEA har riskbedömningen i tabell 8.1 tagits fram.

RPN	Bedömning
60-250	Icke tolererbar risk.
30-59	Begränsad tolererbar risk.
1-29	Tolererbar risk.

Table 8.1: Riskbedömning genom RPN.

Vilka av momenten som analyserades i FMEA måste åtgärdas?

c) Ovanstående riskbedömning använde både OCC, SEV och DET. En alternativ risk-bedömning kan ställas upp enligt tabell 8.2. (IT = Icke tolererbar risk, BT =Begränsad tolererbar risk, T = Tolererbar risk.)

		SEV				
		5	4	3	2	1
OCC	9-10	IT	IT	IT	BT	T
	7-8	IT	IT	BT	BT	T
	5-6	IT	BT	BT	T	T
	3-4	BT	BT	T	T	T
	1-2	T	T	T	T	T

Table 8.2: Riskbedömning genom SEV och OCC.

Vilka av momenten som analyserades i FMEA måste åtgärdas?
Blev det någon skillnad mot resultatet i b)?

Exercise 8.2.
Bilda ett felträd där topphändelsen är att cykeln inte är lämplig att använda och betecknas A. Följande fel ska vara med i felträdet och analyseras i detalj.
B: Cykeln rullar trögt.
C: Ingen bromsverkan.
D: Belysningen uppfyller inte lagkraven.

Exercise 8.3.
Betraka felträdet som beskriver ett bromssystem i figur 18.5 i kompendiet Bergman & Klefsjö. Antag att X, Y och W är oberoende händelser. Sannolikheten att X respektive Y, inträffar under en 10-årsperiod förutsatt en normal service är 0.1%, medan W inträffar endast med 5ppm ($5 \cdot 10^{-6}$) sannolikhet. Vad är sannolikheten att bromsen slutar att fungera under 10-årsperioden?

Exercise 8.4.
Finna alla minimala avbrott i felträdet som är givet i facit till uppgift 8.2.

Part II

Answers

Answers for Chapter 1

Introduction to diagnosis

Exercise 1.1.

a) F_2 and F_3

b) The conclusion is that it can any mode.

c) Either the process is in F_2 or in F_3.

d) The single fault diagnoses are $\{NF, F_1, F_2, F_3\}$.

e) The single fault diagnoses are $\{F_2, F_3\}$.

f) The single fault diagnosis is F_3.

Exercise 1.2.

a)

$$
\begin{aligned}
T_0 =& 2 > 1 \\
T_1 =& 0 < 1 \\
T_2 =& 2 > 1 \\
T_3 =& 2 > 1
\end{aligned}
$$

Tests 0, 2 och 3 reacts.

b) The diagnoses become $A1$ och $M1$. The true mode, $A1$, is a diagnosis, but the diagnosis system can not determine that it must be $A1$ but it can also be $M1$. A closer study of the decision structure shows that the diagnosis system never can separate faults of types $M1$ and $A1$ since $A1$ and $M1$ affect the same set of tests.

Exercise 1.3.

a) The behavior modes are fault free (NF), fault in sensor 1 (F_1), fault in sensor 2 (F_2) and fault in actuator (F_3). The three singular faults can also be combined to produce a further 4 multiple faults.

b) The fault signals can e.g. be introduced as follows:

$$\begin{aligned} \dot{x} &= u + f_3 \\ y_1 &= x + f_1 \\ y_2 &= x + f_2 \end{aligned} \tag{1.1}$$

In NF, $f_1 = f_2 = f_3 = 0$. In F_i, f_i is non zero and the other two fault signals are 0.

Exercise 1.4. Examples on static redundancy are

$$y_1 - y_2 = 0$$

and on temporal redundancy

$$u - \dot{y}_1 = 0$$

Exercise 1.5.

a) Examples of two residuals are

$$\begin{aligned} r_1 &= y_1 - y_2 \\ r_2 &= u - \dot{y}_1 \end{aligned}$$

b) By replacing the the known variables in the residuals with the model equations

$$\begin{aligned} \dot{x} &= u + f_3 \\ y_1 &= x + f_1 \\ y_2 &= x + f_2 \end{aligned}$$

fås

$$\begin{aligned} r_1 &= y_1 - y_2 = (x + f_1) - (x + f_2) = f_1 - f_2 \\ r_2 &= u - \dot{y}_1 = (\dot{x} - f_3) - (\dot{x} + \dot{f}_1) = -\dot{f}_1 - f_3 \end{aligned}$$

c) The decision structure for the residuals in the answer on the (b)-exercise becomes:

	NF	F_1	F_2	F_3
r_1	0	X	X	0
r_2	0	X	0	X

Exercise 1.6.

a)

$$\begin{aligned} |r_1| &= |y_1 - y_2| = |0 - 1| = 1 > 1/2 \quad \to \text{larm} \\ |r_2| &= |u - \dot{y}_1| = |1 - 0| = 1 > 1/2 \quad \to \text{larm} \end{aligned}$$

b) With the residuals from the solution the diagnosis becomes F_1, i.e. sensor 1 is broken. Observe that other selections of residuals can produce other modes as diagnoses. F_1 should however be a diagnosis regardless of which residuals you have selected.

Exercise 1.7. A correct state space model is

$$\begin{aligned}
\dot{\omega} &= -\frac{\mu + f_1}{J}\omega + \frac{k + f_2}{J}u \\
\dot{\varphi} &= \omega \\
y &= \begin{bmatrix} \varphi + f_3 \\ \omega + f_4 \end{bmatrix} \\
\dot{f}_4 &= 0
\end{aligned}$$

where f_i parametrizes fault i. Increased viscous friction gives $f_1 > 0$.

Exercise 1.8. The model becomes

$$\begin{aligned}
J_m\ddot{\varphi}_m &= -F_{v,m}\dot{\varphi}_m + (k_T + f_1)u + \tau_{\text{spring}} \\
\tau_{\text{spring}} &= k(\varphi_a - \varphi_m) + c(\dot{\varphi}_a - \dot{\varphi}_m) \\
(J_a + f_3)\ddot{\varphi}_a &= -\tau_{\text{spring}} + f_4 \\
y &= \varphi_m + \epsilon(f_2) \\
\epsilon(f_2) &= \begin{cases} N(0, \sigma_1^2) & \text{Fault free case} \\ N(0, \sigma_2^2) & \text{Ground wire torn off, } \sigma_2 > \sigma_1 \end{cases}
\end{aligned}$$

where f_i parametrizes fault i.

Exercise 1.9.

a) No redundancy exist. For arbitrary y_1, $\dot{y}_2$ and u values there are a d value so that $d + y_1 + \dot{y}_2 - u = 0$. The model can therefore not be rejected.

b) The relation $y_1 - y_2 + 4u = 0$ shows that static redundancy exist.

c) The relation $2\dot{y}_1 - y_1 - y_2 + 4u = 0$ shows that temporal redundancy exist.

d) The relation $2\dot{y}_1 + 3\dot{u} - y_1 - y_2 + u = 0$ shows that temporal redundancy exist.

e) No redundancy exist.

Exercise 1.10.

b) $P(\text{larm}|NF)$

c) $P(\text{larm}|NF) = 2\phi(-J)$

d) $P(\text{inget larm}|f = f_0)$

e) $P(\text{inget larm}|f = f_0) = \phi(J - f_0) - \phi(-J - f_0)$

f) The false alarms probability is 0 and the probability for a missed detection is 0 for all sizes of the fault $f = f_0 \neq 0$.

Exercise 1.11.

a) $\theta = 0$ and $\theta \in \mathbb{R}$

b) For the case (I) the following applies:

$$\begin{aligned} P(\text{faulty conclusion}|\theta \neq 0) &= P(|T| \leq J|\theta \neq 0) \\ &= P(\text{no alarm}|\theta \neq 0) \end{aligned}$$

i.e. the probability for a faulty conclusion is the same as the probability for a missed detection. Let $p_{MD}(\theta)$ denote the probability for missed detection of a fault $\theta \in]0, \infty[$ and p_{FA} the false alarm probability. The function $p_{MD}(\theta)$ is a monotonic decreasing function in θ where $p_{MD} \to 1 - p_{FA} = 0.99999$ when $\theta \to 0$ and $p_{MD} \to 0$ when $\theta \to \infty$. This means that for small faults it is probable that the wrong conclusion is drawn, but for sufficiently large faults the probability for a faulty conclusion become arbitrarily small. This means that the view I) only is sound if all faults are av sufficient size, i.e. $\theta \neq 0$ implies that $|\theta| > \theta_{min}$ where θ_{min} is sufficiently large.

for the case (II) the probability for a faulty conclusion is zero since the outcome from the test does not say anything about the value of θ. Since no conclusion is drawn we do not know if the diagnosed system is faulty or not, which is the drawback this more cautious view (II).

Exercise 1.12. A simplified model under the assumption $y(t) = 0$ is

$$(b + \Delta b)u = 0$$

When $b + \Delta b \neq 0$ then $u = 0$. The impact on the system from the fault Δb is given by $\Delta b u$ which means that changes in Δb are not visible since $u = 0$. One possible solution is to "exercise" the system once and a while, i.e. excite the system by making small steps with u for finding changes in Δb.

Exercise 1.13.

a)

Observation			Diagnosis
Diode	Light bulb	Switch	
off	off	open/closed	BU
off	lit	open/closed	no diagnoses
lit	off	open	NF,SÖ,LT
lit	off	closed	SÖ,LT
lit	lit	open	SS
lit	lit	closed	NF,SS

b) By changing the desired position of the switch it is possible to uniquely isolate faults NF, SS and BU. The fault modes SÖ and LT can be isolated from the other modes but cannot be separated.

Exercise 1.14.

a) The observation space can, with $z(t) = (u(t), y_w(t), y_p(t))$, be written as

$$\begin{aligned}
\mathcal{O}(NF) &= \{z(t) : \forall t\, W(t) = f(p(t))\, k(u(t)), y_w(t) = W(t), y_p(t) = p(t)\} = \\
&= \{z(t) : \forall t\, y_w(t) = f(y_p(t))\, k(u(t))\} \\
\mathcal{O}(F_a) &= \{z(t) : \exists a\, \forall t\, y_w(t) = f(y_p(t))\, k(u(t))a, a \neq 1\} \\
\mathcal{O}(F_g) &= \{z(t) : \exists g\, \forall t\, y_w(t) = g\, f(y_p(t))\, k(u(t)), g \neq 1\}
\end{aligned}$$

b) It can be seen directly from the answer in the a-exercise that $\mathcal{O}(F_a) = \mathcal{O}(F_g)$ and that the faults thereby not can be isolated from eachother.

for example, assume that the sensors and the actuator has the values $(y_w, y_p, u) = (\hat{y}_w, \hat{y}_p, \hat{u})$ so that

$$\hat{y}_w \neq f(\hat{y}_p)\, k(\hat{u})$$

According to the model

$$\hat{y}_w = k\, f(\hat{y}_p)\, k(\hat{u})$$

where $k = g\, a \neq 1$. From this it is not possible to determine if $a = 1$ and $g = k \neq 1$ or if $g = 1$ and $a = k$, i.e. the faults cannot be isolated from each other.

Answers for Chapter 2

Fault isolation

Exercise 2.1.

a)

	NF	F_1	F_2	F_3
T_1	0	X	X	X
T_2	0	0	X	X
T_3	0	X	0	X
T_4	0	X	X	0

b)

	NF	F_u	F_1	F_2
T_1	0	X	0	X
T_2	0	X	X	0

Exercise 2.2.

a)

$$\begin{aligned} S_1^0 &= \{NF, F_1, F_2, F_3\} & S_1^1 &= \{F_1, F_3\} \\ S_2^0 &= \{NF, F_2, F_3\} & S_2^1 &= \{F_1, F_2\} \\ S_3^0 &= \{NF, F_1, F_2, F_3\} & S_3^1 &= \{F_2\} \end{aligned}$$

b)

$$\begin{aligned} T_1 < J_1 &\Rightarrow S_1 = S_1^0 = \{NF, F_1, F_2, F_3\} \\ T_2 > J_2 &\Rightarrow S_2 = S_2^1 = \{F_1, F_2\} \\ T_3 > J_3 &\Rightarrow S_3 = S_3^1 = \{F_2\} \end{aligned}$$

The diagnosis decision S then becomes:

$$S = \bigcap S_i = \{F_2\}$$

c)

$$\begin{aligned} T_1 > J_1 &\Rightarrow S_1 = S_1^1 = \{F_1, F_3\} \\ T_2 < J_2 &\Rightarrow S_2 = S_2^0 = \{NF, F_2, F_3\} \\ T_3 < J_3 &\Rightarrow S_3 = S_3^0 = \{NF, F_1, F_2, F_3\} \end{aligned}$$

The diagnosis decision S then becomes:

$$S = \bigcap S_i = \{F_3\}$$

Exercise 2.3.

a) In case I $S = \{f_3\}$ and in case II $S = \{f_3\}$.

b) In case I $S = \{f_2\}$ and in case II $S = \{f_1, f_2\}$

c) In case I $S = \{\}$ and in case II $S = \{f_2\}$

d) In case I $S = \{\}$ and in case II $S = \{\}$

e) The diagnoses calculated according to structure I is a subset of the diagnoses calculated according to structure II.

Exercise 2.4. The decision structure defines the following decision S_k^0 and S_k^1:

$$\begin{aligned} S_1^0 &= \{NF, F_1, F_2, F_3\} & S_1^1 &= \{F_2\} \\ S_2^0 &= \{NF, F_1, F_2\} & S_2^1 &= \{F_2, F_3\} \\ S_3^0 &= \{NF, F_1, F_2, F_3\} & S_3^1 &= \{F_1, F_3\} \end{aligned}$$

The conclusions of the singular tests are

$$\begin{aligned} T_1 > J_1 &\Rightarrow S_1 = S_1^1 = \{F_2\} \\ T_2 > J_2 &\Rightarrow S_2 = S_2^1 = \{F_2, F_3\} \\ T_3 < J_3 &\Rightarrow S_3 = S_3^0 = \{NF, F_1, F_2, F_3\} \end{aligned}$$

and the diagnosis decision S then become

$$S = \bigcap S_i = \{F_2\}$$

If the test quantity T_3 goes above the threshold J_3, i.e. $T_3 > J_3$, because of a disturbance the diagnosis decision becomes $S = S_1^1 \cap S_2^1 \cap S_3^1 = \emptyset$.

Exercise 2.5.

a) There exists a fault signal $f_i \neq 0$ so that f_i is a diagnosis but not NF.

b) Assume that we have a fault f_1 i.e. that $f_1 \neq 0$ and $f_2 = f_3 = f_4 = 0$. This means that $r_1 \neq 0$ and $r_2 \neq 0$ i.e. both tests generate alarms. With this test turnout the only diagnosis is f_1 according to the decision structure. Since f_1 is a diagnosis but not NF this means that f_1 is detectable.

c) All faults that affect any of the residuals are detectable, i.e. f_1, f_2 and f_3 are detectable but not f_4.

d) There exists a fault signal $f_i \neq 0$ so that f_i is a diagnosis, but not f_j.

e) If $f_2 \neq 0$ and $f_1 = f_3 = f_4 = 0$ then $r_1 \neq 0$ and $r_2 = 0$. The diagnoses are then f_1 and f_2. This means that f_2 is isolable from f_3 and f_4 but not from f_1.

f) The detectable faults are f_1, f_2 and f_3. The isolablity is specified in the following table:

f_1 is isolable from f_2, f_3 and f_4
f_2 is isolable from f_3 and f_4
f_3 is isolable from f_2 and f_4
f_4 is not isolable from any other fault.

The isolability matrix becomes

	f_1	f_2	f_3	f_4
f_1	X	0	0	0
f_2	X	X	0	0
f_3	X	0	X	0
f_4	X	X	X	X

Exercise 2.6.

a) Let $z = (u, y_1, y_2)'$. Define

$$O(NF) = \{z | \exists x \, \dot{x} = u \wedge y_1 = x \wedge y_2 = x\}$$

and

$$O(f_1) = \{z | \exists x \exists f_1 \, \dot{x} = u \wedge y_1 = x + f_1 \wedge y_2 = x\}$$

The corresponding definitions can be made for faults f_2 and f_3. Fault f_i is detectable in the model if

$$O(f_i) \setminus O(NF) \neq \emptyset$$

According to the definition all faults are detectable.

b) A fault f_i is isolable from a fault f_j if

$$O(f_i) \setminus O(f_j) \neq \emptyset$$

According to the definition all faults are uniquely isolable, i.e. the isolability matrix is

	f_1	f_2	f_3
f_1	X	0	0
f_2	0	X	0
f_3	0	0	X

c) All faults are detectable with the diagnosis system. So there is no difference between the detectability of the diagnosis system and the models. The isolability of the diagnosis system is given by

	f_1	f_2	f_3
f_1	X	0	0
f_2	X	X	0
f_3	X	0	X

The difference between the isolability of the diagnosis system and the models is that f_2 and f_3 are not isolable from f_1 with the diagnosis system.

d) Making f_2 and f_3 isolable from f_1 with a residual r_3 requires that the residual reacts on f_2 and f_3 but not on f_1. The expanded decision structure becomes

	NF	f_1	f_2	f_3
r_1	0	X	X	0
r_2	0	X	0	X
r_3	0	0	X	X

e) For creating a residual that is not sensitive to f_1, f_1 is considered an unknown variable, i.e. f_1 is decoupled. This gives that the residual becomes

$$r_3 = \dot{y}_2 - u = \dot{f}_2 + f_3$$

where the internal form shows that the residual has the desired fault sensitivity.

Exercise 2.7. The requirements for detectability gives that all columns in the decision structure must be non zero. The zeros in the isolability matrix puts the following demands on the decision structure:

position	NF	f_1	f_2	f_3
$(1,2)$	0	X	0	?
$(1,3)$	0	X	?	0
$(2,1)$	0	0	X	?
$(3,1)$	0	0	?	X

a) The following decision structure fulfills the requirements:

	NF	f_1	f_2	f_3
r_1	0	X	0	0
r_2	0	0	X	X

the detectability and isolability becomes exactly the specified.

b) Under the assumption that only a single fault at the time can be decoupled the following three residuals are required:

	NF	f_1	f_2	f_3
r_1	0	X	0	X
r_2	0	X	X	0
r_3	0	0	X	X

The decision structure gives full detectability and singular fault isolability, which is more than required by the specification. All tests are required in the sense that if any test is removed the specification will not be fulfilled.

Exercise 2.8. Besides d the following signals should be decoupled for the four cases:

- 1: no signal is decoupled
- 2: f_u is decoupled
- 3: f_1 is decoupled
- 4: f_2 is decoupled

Exercise 2.9. Both should be decoupled.

Exercise 2.10.

a) A fault $f_u(t)$ affects the system in the same way as $f_a(t) = bf_u(t)$. The faults cannot be isolated from each other.

b) Fault f_a cannot be isolated from f_u, since an arbitrary fault $f_a(t) = \Delta a_0 x(t)$ affects the system in the same way as $f_u(t) = \Delta a_0 x(t)/b$. Fault f_u can be isolated from f_a, since we can create a residual that decouples f_a but that is sensitive to f_u in the following way:

$$\begin{aligned} r_1(t) &= y(t-2)(y(t) - bu(t-1)) - y(t-1)(y(t-1) - bu(t-2)) \\ &= b(y(t-2)f_u(t-1) - y(t-1)f_u(t-2)) \end{aligned}$$

c) The faults can be isolated from each other. Residual r_1 in the solution to exercise (b) can isolate f_u from f_a and the residual

$$\begin{aligned} r_2(t) &= y(t) - y(t-1) - a(y(t-1) - y(t-2)) - b(u(t-1) - u(t-2)) \\ &= \Delta a(y(t-1) - y(t-2)) \end{aligned}$$

is sensitive to f_a but decouples f_u and can therefore isolate f_a from f_u.

Exercise 2.11. A diagnosis system is "complete" if $F_p = F_j$ entails that F_j is a diagnosis, i.e. that $F_j \in S = \bigcap_{k=1}^{n} S_k$ where n is the number of tests in the diagnosis system. The trick is therefore to show that $F_j \in S_k$ for all $k = 1, \ldots n$. Assume that the system is in mod F_j, i.e. $F_p = F_j$. Consider the column k in the decision structure s_{kj}, $k = 1, \ldots, n$ that corresponds to F_j. The tests can be divided into three cases: 1) $s_{kj} = 0$, 2) $s_{kj} = X$ and 3) $s_{kj} = 1$.

1) Consider an arbitrary test k so that $s_{kj} = 0$. According to (3.12) the test k does not generate an alarm. Since $s_{kj} = 0$ it follows that $F_j \in S_k$.

2) Consider an arbitrary test k so that $s_{kj} = X$. Regardless if the test generates an alarm or not it hold that $F_j \in S_k$.

3) Consider an arbitrary test k so that $s_{kj} = 1$. According to (3.12) the test k generates an alarm. Since $s_{kj} = 1$ it follows that $F_j \in S_k$.

Since all three cases leads to $F_j \in S_k$ it means that none of the tests rejects F_j and therefore it holds that $F_j =\in \bigcap_{k=1}^{n} S_k = S$.

Answers for Chapter 3

Design of Test Quantities

Exercise 3.1.

a) The probability of false alarm $P(\text{alarm}|NF) = P(|r| > J|f = 0)$ and probability of missed detection for a fault with size $f = f_0$ is $P(\text{no alarm}|f = f_0) = P(|r| \leq J|f = f_0)$.

b) 1.

$$\begin{aligned}P(\text{alarm}|NF) &= P(|r| > J|f = 0)\\ &= P(r/\sigma < -J/\sigma|f = 0) + P(r/\sigma > J/\sigma|f = 0)\\ &= \Phi(-J/\sigma) + (1 - \Phi(J/\sigma)) = 2\Phi(-J/\sigma)\end{aligned}$$

2.

$$\begin{aligned}P(\text{no alarm}|f = f_0) &= P(|r| \leq J|f = f_0)\\ &= P(-J \leq r \leq J|f = f_0)\\ &= P((-J - f_0)/\sigma \leq (r - f_0)/\sigma \leq (J - f_0)/\sigma|f = f_0)\\ &= \Phi((J - f_0)/\sigma) - \Phi((-J - f_0)/\sigma)\end{aligned}$$

3.

$$\begin{aligned}P(|r| > J|f = 0) = 2\Phi(-J/\sigma) &= \alpha \Leftrightarrow\\ -J/\sigma = \Phi^{-1}(\alpha/2) &\Leftrightarrow\\ J = -\sigma\Gamma(\alpha/2)&\end{aligned}$$

c) The probability of false alarm decreases and the probability of missed detection increases with increasing value on the threshold. When the threshold is 0 then the probability for false alarm is 1 and the probability of missed detection is 0. When the threshold goes to infinity then the probability for false alarm is 0 and the probability of missed detection is 1.

d)

$$\begin{aligned}\beta(f_0) &= P(\text{alarm}|f = f_0)\\ &= P(|r| > J|f = f_0)\\ &= P((r - f_0)/\sigma < (-J - f_0)/\sigma|f = f_0) + P((r - f_0)/\sigma > (J - f_0)/\sigma|f = f_0)\\ &= \Phi((-J - f_0)/\sigma) + (1 - \Phi((J - f_0)/\sigma))\end{aligned}$$

Exercise 3.2.

a)
```
sigma = 2;
alpha = 0.01;

J = -sigma*norminv(alpha/2);
```

b)
```
f = [0:0.1:15];
beta = normcdf(-J,f,sigma) + (1-normcdf(J,f,sigma));
plot(f,beta)
```

c)
```
f0 = 5
J = [0:0.1:100];
pfa = 2*normcdf(-J,0,sigma);
pd  = normcdf(-J,f0,sigma) + (1-normcdf(J,f0,sigma))
plot(pfa,pd)
```

Exercise 3.3. Let $p_d(J)$ be the probability for detection and let $p_{fa}(J)$ be the probability for false alarm. Death of the crew can either happen if the fault is not detected or if the test detects the fault and the crew dies during the escape from the space shuttle, that is

$$\begin{aligned}P(\text{death}) &= P(\neg\text{alarm}|\text{fault})p_1 + P(\text{alarm})p_2\\ &= (1 - p_d(J))p_1 + P(\text{alarm})p_2\end{aligned} \tag{3.1}$$

The probability for alarm can be expressed as

$$\begin{aligned}P(\text{alarm}) &= P(\text{alarm}|\text{NF})(1 - p_1) + P(\text{alarm}|\text{fault})p_1\\ &= p_{fa}(J)(1 - p_1) + p_d(J)p_1\end{aligned}$$

Using (3.1) gives $P(\text{death})$ as a function of threshold. Example of code that generates the results are:

```
J = [-10:0.01:10];
f = 5;
p1 = 0.01;
p2 = 0.005;
pfa = 1-normcdf(J,0,2);
pd = 1-normcdf(J,f,2);
p = (1-pd)*p1 + (pfa*(1-p1)+pd*p1)*p2;
```

```
plot(J,p)
min(p)
J(find(min(p)==p))
```

The minimum is achieved for $J = 1.94$ and the probability that the crew will be killed is reduced from 0.01 to 0.0015.

Exercise 3.4.

a) The test quantity is

$$T_1 = \min_{\theta_1} \frac{1}{\sigma^2} \sum_{i=1}^{N} (y(t) - u_1(t)\theta_1)^2$$

and when the null hypothesis is true then is T_1 χ^2-distributed with $N-1$ degrees of freedom. Let $F_{\chi^2(n)}$ be the cumulative distribution function for a χ^2-distribution with n degrees of freedom. Then

$$J_1 = F^{-1}_{\chi^2(N-1)}(1 - p_{fa})$$

is the threshold that results in $P(T_1 > J_1|H_0) = p_{fa}$.

b) If $r(i) = y(i) - u(i)\theta$ and $f(r(i)|\theta)$ distribution function for $r(i)$ given θ then the following holds

$$T_2 = \max_{\theta_2,\theta_3} \ln \prod_{i=1}^{N} f(r(i)| \begin{bmatrix} 0 & \theta_2 & \theta_3 \end{bmatrix}^T)$$

By using f as density function for $N(0, \sigma)$ and after simplification you get the following

$$T_2 = \min_{\theta_2,\theta_3} \frac{1}{\sigma^2} \sum_{i=1}^{N} (y(i) - u_2(i)\theta_2 - u_3(i)\theta_3)^2$$

which is the same results the prediction error method would give. During the simplification the constant terms have been neglected and the test quantity has been scaled. The test quantity is χ^2-distributed with $N-2$ degrees of freedom when the null hypothesis is true. The threshold is the same as in exercise (a).

$$J_2 = F^{-1}_{\chi^2(N-2)}(1 - p_{fa})$$

c) A test quantity of the parameter estimation is

$$T_3 = |\hat{\theta}_2 - \theta_{2,nom}| = |\arg_{\theta_2} \min_{\theta} \sum_{i=1}^{N} (y(i) - u(i)\theta)^2 - 0|$$

d)

	NF	F_1	F_2	F_3
T_1	0	0	X	X
T_2	0	X	0	0
T_3	0	0	X	0

Exercise 3.5.

a) Equation (3.1) can be written as

$$y(t) = \varphi(t)\theta + v(t) = [u(t)\ 1] \begin{pmatrix} a \\ b \end{pmatrix} + v(t)$$

and this gives the following matrices:

$$Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \qquad \Phi = \begin{bmatrix} u(1) & 1 \\ \vdots & \vdots \\ u(N) & 1 \end{bmatrix} \qquad \theta = \begin{bmatrix} a \\ b \end{bmatrix} \qquad V = \begin{bmatrix} v(1) \\ \vdots \\ v(N) \end{bmatrix}$$

b) The function that will be minimized can be written as

$$f(\theta) = |Y - \Phi\theta|^2$$

From the fundamentals from linear algebra we know that a minimizing θ is given by the solution of the equation

$$\Phi^T(Y - \Phi\theta) = 0$$

If we have excitation, that is, if the matrix $\Phi^T\Phi$ is invertible, then the minimization problem has a unique optimum in

$$\hat{\theta} = (\Phi^T\,\Phi)^{-1}\,\Phi^T\,Y$$

Explicit calculations gives

$$\Phi^T\Phi = \begin{pmatrix} \sum_t u^2(t) & \sum_t u(t) \\ \sum_t u(t) & N \end{pmatrix}$$

whose determinant is

$$N\sum_t u^2(t) - (\sum_t u(t))^2$$

Let $\bar{u}$ be the mean value of the signal u, the determinant can then be written as

$$N\sum_t u^2(t) - (\sum_t u(t))^2 = N\sum_t \left(u^2(t) - \bar{u}^2\right)$$

This expression is 0 if and only if $u(t) = \bar{u}$ for all t, that is, the input signal u is constant.

c) The distribution of $T = KV$ is

$$\begin{aligned} &E\{T\} = KE\{V\} = 0 \\ &cov(T) = E\{TT^T\} = KE\{VV^T\}K^T = K\Sigma K^T \end{aligned} \tag{3.2}$$

where Σ is the co-variance for V, that is, $\Sigma = E\{VV^T\}$. The estimation (3.3) become, after a substitution of Y of the right side in (3.2)

$$\hat{\theta} = \theta + (\Phi^T\Phi)^{-1}\Phi^T V$$

Since $\hat{\theta}$ is a linear combination of normal distributed variables then V is $\hat{\theta}$ normal distributed. Using the results in (3.2) with $K = (\Phi^T\Phi)^{-1}\Phi^T$ it follows that $E\{\hat{\theta}\} = \theta$, that is, $\hat{\theta}$ is unbiased and the covariance matrix becomes

$$\Sigma_{\hat{\theta}} = (\Phi^T\Phi)^{-1}\sigma_v^2$$

where we have used $(\Phi^T\Phi)^{-T} = (\Phi^T\Phi)^{-1}$.

Exercise 3.6.

a) If $Y = [y(1), \ldots, y(N)]^T$, $U_1 = [u_1(1), \ldots, u_1(N)]^T$ and $V = [v(1), \ldots, v(N)]^T$ then the model is valid given H_0:

$$Y = U_1\theta_1 + V$$

The parameter θ_1 that minimizes the test quantity is given be the solution from the regression

$$\hat{\theta}_1 = (U_1^T U_1)^{-1} U_1^T Y$$

By formulating the residual $R = Y - U_1\hat{\theta}_1$ the test quantity can be written as

$$T_1 = \frac{1}{\sigma^2} R^T R$$

b) Låt

$$U_{23} = \begin{bmatrix} u_2(1) & u_3(1) \\ \vdots & \vdots \\ u_2(N) & u_3(N) \end{bmatrix}$$

Then T_2 is given by

$$\begin{bmatrix} \hat{\theta}_2 \\ \hat{\theta}_3 \end{bmatrix} = (U_{23}^T U_{23})^{-1} U_{23}^T Y$$

$$R = Y - U_{23} \begin{bmatrix} \hat{\theta}_2 \\ \hat{\theta}_3 \end{bmatrix}$$

$$T_2 = \frac{1}{\sigma^2} R^T R$$

c) Låt $U = [u(1)^T, \ldots, u(N)^T]^T$. The model that is used for the parameter estimation is then:

$$Y = U\theta + V$$

The estimation is given by

$$\hat{\theta} = (U^T U)^{-1} U^T Y$$

and the test quantity is thus $T_3 = \hat{\theta}_2$.

From Exercise 3.5 (c) we get

$$\hat{\theta} \sim N(\theta, (U^T U)^{-1}\sigma^2)$$

Furhter, $T_3 = K\hat{\theta}$ where $K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. Then $T \sim N(\theta_2, \bar{\sigma})$, where $\bar{\sigma}^2 = K(U^TU)^{-1}K^T\sigma^2$. Let $F_{N(\mu,\sigma)}(x)$ be the cumulative distribution function for $N(\mu, \sigma)$. The threshold that satisfies $P(|T_3| > J_3|H_0) = p_{fa}$ is given by

$$J_3 = F^{-1}_{N(0,\bar{\sigma})}(1 - p_{fa}/2)$$

Exercise 3.7.

a)
```
% The test quantity as defined in Exercise a) will be computed
theta1 = U(:,1)\Y;
R1 = Y-U(:,1)*theta1;
T1 = R1'*R1;
J1 = chi2inv(1-pfa,N-1);
T1 = T1/J1;

% The test quantity as defined in Exercise b) will be computed
theta23 = U(:,[2 3])\Y;
R2 = Y-U(:,[2 3])*theta23;
T2 = R2'*R2;
J2 = chi2inv(1-pfa,N-2);
T2 = T2/J2;

% The test quantity as defined in Exercise c) will be computed
theta123 = U\Y;
T3 = theta123(2);
Phi = U;
K = [0 1 0];
sigma = sqrt(K*inv(Phi'*Phi)*K');
J3 = norminv(1-pfa/2,0,sigma);
T3 = abs(T3)/J3;
```

b) The following matrix has same size as the decision structure and should have values close to p_{fa} in those positions where there are zeros in the decision structure and values close to 1 in the rest of the positions.

```
pd = [sum(results.NF.A)' sum(results.F1.A)'...
      sum(results.F2.A)' sum(results.F3.A)']/M;
```

Exercise 3.8.

a) Let $Y = \begin{bmatrix} y(1) & y(2) & \dots & y(N) \end{bmatrix}^T$ and $U_i = \begin{bmatrix} u_i(1)^T & u_i(2)^T & \dots & u_i(N)^T \end{bmatrix}^T$ för $i = 1, 2, 3$. According to the assumption that the vectors U_i are linear independent. The prediction

$$\hat{Y} = \hat{\theta}_1 U_1 + \hat{\theta}_2 U_2$$

is the orthogonal projection of Y in the plane Ω that is spanned by U_1 and U_2. If $\theta_3 = 0$ then will Y be included or near the plane and the estimations of θ_1 and θ_2 will become good. This means that the test quantity if sensitive for F_1 but decouples F_2. If $\theta_3 \neq 0$ then will Y not be in the included in the plane Ω. The estimation of θ_1 will be zero only if U_1 and U_3 are orthogonal. There is no gDet finns det ingen garanti för, så teststorheten kan vara känslig föruarantee that the test quantity will be sensitive to F_3.

b)

$$H_0 : F_p \in \{NF, F_2\} \qquad H_1 : F_p \in \{F_1, F_3\}$$

c)

	NF	F_1	F_2	F_3
T	0	X	0	X

Exercise 3.9. According to the figure $\delta > \|\Delta G(s)\|$ approximate -20 dB, that is

$$\delta = 10^{-20/20} = 0.1$$

A residual generator can be written for example as

$$r = y - \frac{1}{s+1}u$$

Using y when f is zero gives

$$r = \Delta G(s)u$$

Using the approximation $\delta > \|\Delta G(s)\|$ yields an adaptive threshold as

$$J_{adap} = \delta|u| + J_0 = 0.1|u| + J_0$$

Exercise 3.10.

a)

$$P(FA) = \beta(\theta_0) = \beta(1)$$

b)

$$P(MD|\theta \neq 1) = 1 - \beta(\theta)$$

Exercise 3.11.

a)

$$\begin{aligned}\Sigma_x = E\{x(t)x(t)^T\} = \\ = E\{(Ax(t-1) + Bn(t-1)(Ax(t-1) + Bn(t-1))^T\} = \\ = A\Sigma_x A^T + B\Sigma_n B^T\end{aligned}$$

The mixed terms becomes 0 since $E\{x(t)n(t)\} = 0$, that is

$$\Sigma_x = A\Sigma_x A^T + B\Sigma_n B^T$$

and symmetrical solution of the equation is guaranteed if A is stable.

b)

$$\Sigma_y = E\{y(t)y^T(t)\} = C\Sigma_x C^T$$

c)
$$\Sigma_y = E\{y(t)y^T(t)\} = C\Sigma_x C^T + D\Sigma_n D^T$$

d) Since x is a stationary stochastic process the following holds:

$$\Sigma_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_x(\omega)d\omega = \int_{-\infty}^{\infty}(j\omega I - A)^{-1}B\Sigma_n B^T(j\omega I - A)^{-T}d\omega =$$
$$= /^*\text{ parseval}^*/ = \int_0^{\infty} e^{At}B\Sigma_n B^T e^{A^T t}dt$$

Multiplication from the left with A and integration by parts gives:

$$A\Sigma_x = \int_0^{\infty} Ae^{At}B\Sigma_n B^T e^{A^T t}dt =$$
$$= [e^{At}B\Sigma_n B^T e^{A^T t}]_0^{\infty} - \int_0^{\infty} e^{At}B\Sigma_n B^T e^{A^T t}dtA^T =$$
$$= /^*\ A\text{ stable}^*/ = -B\Sigma_n B^T - \Sigma_x A^T$$

Thus, the following holds

$$A\Sigma_x + \Sigma_x A^T + B\Sigma_n B^T = 0$$

symmetrical solution of the equation is guaranteed if A is stable.

Exercise 3.12.

a)
$$r_v(k) = \begin{cases}\sigma_v^2 & k = 0\\ 0 & k \neq 0\end{cases}$$

b) There is no dependency between $v(t)$ and $v(t-k)$ when $k \neq 0$.

Exercise 3.13.

a) Find J such that $\alpha = P(|T| > J|\theta = 0)$.

```
J = norminv(1-0.05/2,0,0.7)
```

b)
```
theta = [-5:0.1:5]';
beta = 1-normcdf(J,theta,0.7)+...
          normcdf(-J,theta,0.7);
plot( theta,beta )
```

Exercise 3.14.

a)
```
% Monte-Carlo simulation to determine
% the power function.
J = 1.372
n = 1e5; % Number of simulations
Sigma = [0.3,-0.2;-.2,1]; theta = -5:0.1:5;
for k = 1:2
  for j = 1:length(theta)
```

```
    if k == 1
      my1 = theta(j); my2 = 0;
    else
      my1 = 0; my2 = theta(j);
    end
    my = [my1, my2];
    R = mvnrnd(my,Sigma,n);     % Random number
    X = find(abs(R(:,k)) > J); % |T| > J
    beta(k,j) = length(X)/n;
  end
end
figure(1);clf; plot(theta,beta)
hold on; plot([J,J],[0,1],':r'); plot([-J,-J],[0,1],':r')
% For comparison
beta1=1-normcdf(J,theta,Sigma(1,1))+normcdf(-J,theta,Sigma(1,1));
beta2=1-normcdf(J,theta,Sigma(2,2))+normcdf(-J,theta,Sigma(2,2));
plot(theta,beta1,'b.'); plot(theta,beta2,'g.')
```

b) It means roughly that if an estimation gets to big due to noise then it tends to "pull" also the other estimation. The area $|\theta| = J$ corresponds to a circle, but the area of significance corresponds to an ellipse which gives difference levels of significance in different directions. .

c) The null hypothesis is that $\theta_1 = 0$ and $\theta_2 = 0$. The test cannot be used for isolation of faults between fault $\theta_1 \neq 0$ and fault $\theta_2 \neq 0$.

d) The covariance for KT is $\text{cov}(KT) = K\Sigma K^T$. The goal is to now find a K such that $K\Sigma K^T$ is a diagonally matrix with equal elements in the diagonal. One way to find K is by using SVD of Σ, that is

$$\Sigma = USU^T$$

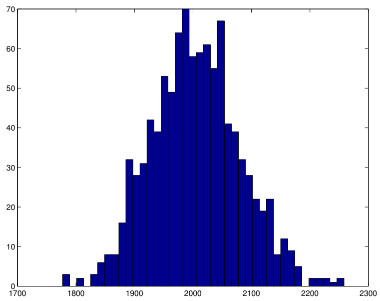
where U is an orthogonal matrix and S is a non-singular (since Σ is non-singular) diagonal matrix with its singular values (> 0) in the diagonal. Chosse for example $K = S^{-1/2}U^T$ which gives

$$K\Sigma K^T = S^{-1/2}U^TUSU^TUS^{-1/2} = S^{-1/2}SS^{-1/2} = I$$

Exercise 3.15. Example of code in Matlab:

```
N = 1000;
theta = 1;
T = zeros(N,1);
for l=1:N
  y = randn(1000,1)+theta*ones(1000,1);
  T(l) = y'*y;
end
hist(T,40)
```

For $\theta = 1$ the histogram becomes



Exercise 3.16.

a) Used notation: $\alpha = P(T > J|H_0) = 0.0005$, $\beta = P(H_0) = 0.9999$.

$$P(H_0|T > J) = \frac{P(T > J|H_0)P(H_0)}{P(H_0)P(T > J|H_0) + P(H_1)P(T > J|H_1)} =$$
$$= \frac{\alpha\beta}{\alpha\beta + \underbrace{(1-\alpha)}_{\text{ty } J=0.5}(1-\beta)} = \frac{1}{1 + \frac{(1-\alpha)(1-\beta)}{\alpha\beta}} \approx 0.83$$

b)

$$P(H_0|T_1 > 0.5 \wedge T_2 > 0.5) = \frac{1}{1 + \frac{P(T_1>J|H_1)P(T_2>J|H_1)P(H_1)}{P(T_1>J|H_0)P(T_2>J|H_0)P(H_0)}} =$$
$$= \frac{1}{1 + \frac{(1-\beta)(1-\alpha)^2}{\beta\alpha^2}} \approx 0.0025 > \alpha$$

c) The general expression for n independent samples is

$$\frac{1}{1 + \frac{(1-\beta)(1-\alpha)^n}{\beta\alpha^n}}$$

With some testing it can be seen that $n = 3$ is enough.

Exercise 3.17.

a)

$$y(t) = x(t) + v(t) = ax(t-1) + bu(t-1) + v(t) = [y(t-1)\ u(t-1)] \begin{bmatrix} a \\ b \end{bmatrix} + w(t)$$

where $w(t) = v(t) - av(t-1)$.

b) No, since

$$E\{w(t)w(t-1)\} = E\{(v(t) - av(t-1))(v(t-1) - av(t-2))\} = -a\sigma_v^2 \neq 0$$

c) Since the noise is correlated the estimate will be biased, that is, regardless how much data we collect it is impossible to estimate the true θ. This can be seen by the following calculations. Let W be a vector with all $w(t)$ on top of each other, then the following holds (only use the expression of $\hat{\theta}$ to verify):

$$E\{\Phi^T\Phi(\hat{\theta}-\theta_0)\} = E\{\Phi^T W\}$$

The expression $E\{\Phi^T W\}$ thus tells if the estimate will be biased or not. Using it gives

$$E\{\Phi^T W\} = E\{\begin{bmatrix} y(1) & \dots & y(N-1) \\ u(1) & \dots & u(N-1) \end{bmatrix} \begin{bmatrix} w(2) \\ \vdots \\ w(N) \end{bmatrix}\} =$$

$$= E\{\begin{bmatrix} \sum_{i=1}^{N-1} y(i)w(i+1) \\ \sum_{i=1}^{N-1} u(i)w(i+1) \end{bmatrix}\} = \begin{bmatrix} (N-1)E\{y(i)w(i+1)\} \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} (N-1)E\{y(i)(v(i+1)-av(i))\} \\ 0 \end{bmatrix} = \begin{bmatrix} -a(N-1)\sigma_v^2 \\ 0 \end{bmatrix} \neq 0$$

which means that the estimate will not converge.

$$E\{\hat{\theta}\} = \theta_0 + (\Phi^T\Phi)^{-1} \begin{bmatrix} -a(N-1)\sigma_v^2 \\ 0 \end{bmatrix} = \theta_0 + \text{bias}$$

Exercise 3.18.

a)

```
N1 = size(u,1);
z1 = [y u];

Y1 = z1(3:N1,1);
phi1 = [z1(2:N1-1,2) z1(1:N1-2,2)];
thls1 = inv(phi1'*phi1)*phi1'*Y1;

plot( [z1(3:N1,1) phi1*thls1])
```

b)

```
N2 = size(u,1);
z2 = [y u];

Y2 = z2(3:N2,1);
phi2 = [z2(2:N2-1,2) z2(1:N2-2,2)];
thls2 = inv(phi2'*phi2)*phi2'*Y2;
plot( [z2(3:N2,1) phi2*thls2])
```

c) Yes, there is a difference. It depends on the excitation of the system. In the other case then $u = 1$ for all t which leads to that it is impossible to estimate both b_1 and b_2.

d) By varying the input signal u then the data will be informative enough to be able to estimate both parameters correctly.

e) The prediction error method works well even if the estimates of θ are bad.

Answers for Chapter 4

Linear Residual Generation

Exercise 4.1.

a)

$$G_u(p) = \frac{1}{p+a}, \quad G_f(p) = 1$$

b)

$$0 = \begin{bmatrix} -(p+a) \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f = H(p)x + L(p)z + F(p)f$$

Exercise 4.2.

a) The observation set is given by

$$\mathcal{O} = \{(y,u) | \dot{y} + ay - u = 0\}$$

b)

$$\mathcal{O} = \{(y(t), u(t)) | \exists y_0 \; y(t) = y_0 e^{-at} + \int_0^t e^{-a(\tau - t)} u(\tau) d\tau\}$$

c)

$$\dot{y} + ay - u = 0$$

d) Add dynamics, for example calculate the residual as follows

$$\dot{r} + \beta r = \dot{y} + ay - u, \quad \beta > 0$$

in state-space form becomes

$$\begin{aligned} \dot{w} &= -\beta w + (a - \beta)y - u \\ r &= w + y \end{aligned}$$

Exercise 4.3.

a) Calculation of the expressions for $y(t)$ and $\hat{y}(t)$ given $z \in \mathcal{O}$ becomes

$$\hat{y}(t) = \hat{y}_0 e^{-t} + \int_0^t e^{-(t-\tau)} u(\tau) d\tau$$
$$y(t) = y_0 e^{-t} + \int_0^t e^{-(t-\tau)} u(\tau) d\tau$$

The residual is then

$$r(t) = y(t) - \hat{y}(t) = (y_0 - \hat{y}_0) e^{-t}$$

which results in that the residual generator satisfies the definition.

b) IF $a = -1$ the expression becomes

$$r(t) = y(t) - \hat{y}(t) = (y_0 - \hat{y}_0) e^{t}$$

which, since the initial condition is unknown, yields that the filter is not a residual generator.

c) If $a = -1$ then is the consistency relation

$$\dot{y} - y - u = 0$$

Add dynamics for the residual generator and generate the residual according to

$$\dot{r} + \beta r = \dot{y} - y - u, \quad \beta > 0$$

which on state-space form is

$$\dot{w} = -\beta w - (1 + \beta) y - u$$
$$r = w + y$$

This expression can be shown that it fulfills the definition of residual generators.

Exercise 4.4.

a) The consistency relation and the residual generator:

$$y_1 + 3y_2 + 6u = 0 \qquad r = y_1 + 3y_2 + 6u$$

b) The internal form of the residual generator becomes

$$r = y_1 + 3y_2 + 6u = f$$

c)

$$M = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian elimination gives

$$M \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & -3 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 3 & 6 \end{bmatrix}$$

The bottom line means that

$$y_1 + 3y_2 + 6u = 0$$

d) All sequences with $\{x_1, x_2, d\} > \{y_1, y_2, u\}$ would work.

Exercise 4.5.

a) Hint: Remember the writing

$$y = \frac{b(p)}{a(p)} u$$

which means that the two time-signals $y(t)$ and $u(t)$ satisfies the differential equation

$$a(p)y(t) - b(p)u(t) = 0$$

b) $\mathcal{O}_2 \subseteq \mathcal{O}_1$

Exercise 4.6.

a) A first order consistency relation is

$$(p + 1)y - u = 0$$

A residual with wanted dynamic properties is generated by the differential equation

$$\dot{r} + \alpha r = (p + 1)y - u$$

The transfer function for the residual generator can then be written as:

$$r = R(s)z = \frac{1}{s + \alpha} \begin{bmatrix} s + 1 & -1 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

and with the state $w = y - r$ yields the following state-space representation

$$\begin{aligned} \dot{w} &= -\alpha w + (\alpha - 1)y + u \\ r &= -w + y \end{aligned}$$

b) A state-space representation of the system is

$$\begin{aligned} \dot{x} &= -x + u \\ y &= x \end{aligned}$$

c) A residual generator based on estimation of states is given by

$$\begin{aligned}\dot{\hat{x}} &= -\hat{x} + u + K(y - \hat{x})\\ r &= y - \hat{x}\end{aligned}$$

where the observer gain K is chosen so the observer becomes stable.

d) By a simple rewriting of the solution in the observer case yields the following

$$\begin{aligned}\dot{\hat{x}} &= -(1+K)\hat{x} + Ky + u\\ r &= -\hat{x} + y\end{aligned}$$

where it can be seen directly that with $K = \alpha - 1$ the two solution becomes identical.

e) For example when we have disturbances d that shall be decoupled or if the residual generator does not have the same order as the system

Exercise 4.7.

a) Transfer matrix

$$y = \frac{p+1}{2p+1}u$$

and state-space form

$$\begin{aligned}\dot{x} &= -\frac{1}{2}x + \frac{1}{2}u\\ y &= \frac{x+u}{2}\end{aligned}$$

b) Transfer matrix

$$y = \begin{bmatrix}\frac{1}{p+1}\\ \frac{1}{p+2}\end{bmatrix} u$$

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix} u = Ax + B_u u\\ y &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} x = Cx\end{aligned}$$

c)

$$\begin{aligned}\dot{x} &= -\frac{1}{2}x + \frac{1}{2}u\\ y &= \frac{x+u}{2} + f\end{aligned}$$

d)

$$\begin{aligned}\dot{x} &= Ax + B_u u + \begin{bmatrix}B_u^1 & 0\\ 0 & B_u^2\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix}\\ y &= Cx + \begin{bmatrix}1\\ 0\end{bmatrix} d\end{aligned}$$

e) With the given time constant then the dominant pole should be close to $s = -3$. One, of many, such residual generator is

$$r = \frac{p+2}{p+3} y_2 - \frac{1}{p+3} u$$

With, for example, the state-variable $z = y_2 - r$ the following state-space form is achieved

$$\begin{aligned} \dot{z} &= -3z + y_2 + u \\ r &= -z + y_2 \end{aligned}$$

Exercise 4.8.

a)

$$\dot{y} + y - u - \dot{f} - f = 0$$

b)

$$r = \dot{y} + y - u$$

c)

$$r = \dot{y} + y - u = \dot{f} + f$$

d) In the frequency domain:

$$r = \frac{1}{s+\alpha} [s+1 \quad -1] \begin{pmatrix} y \\ u \end{pmatrix} = \frac{s+1}{s+\alpha} f$$

On state-space form the residual generator is the following

$$\begin{aligned} \dot{z} &= -\alpha z + [1-\alpha \quad -1] \begin{pmatrix} y \\ u \end{pmatrix} \\ r &= z + [1 \;\; 0] \begin{pmatrix} y \\ u \end{pmatrix} \end{aligned}$$

In the time domain the calculation form is given by the following differential equation

$$\dot{r} + \alpha r - \dot{y} - y + u = 0$$

and the internal form

$$\dot{r} + \alpha r - \dot{f} - f = 0$$

Exercise 4.9.

a) The polynomial $a(p)$ needs to be of at least of order 2 and all zeros to $a(s)$ must be in the left half-plane.

b) See definition on observable canonical form.

Exercise 4.10.

a)

$$\omega - s\varphi = 0$$

b) There exists two linear independent relations. The reason for this is that we have an exactly determined model and two sensors which results in a redundancy of two for the whole model.

Exercise 4.11.

$$\begin{aligned} y_2 - y_3 &= 0 \\ y_2(1 + pT) - y_1 &= 0 \end{aligned}$$

and all filtered linear combinations of the two.

Hint regarding other than the above mentioned that would be of interest: Think about possibilities for fault isolation.

Exercise 4.12.

a)

	f_1	f_2
r	X	0

b)

$$0 = \begin{bmatrix} -(p+1) & 0 & 0 \\ 1 & -p & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ f_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} f_1$$

Exercise 4.13.

a) No, it is not possible to isolate the faults from eachother.
Hint: What happens if f_1 is decoupled?

b) No, it is possible to isolate the faults.
How many linearly independent signals can be decoupled in each residual?

c) We can isolate an unlimited number of faults f_i if $B_i := \begin{bmatrix} B_f(:,i) \\ D_f(:,i) \end{bmatrix}$ has properties as in b).

Exercise 4.14.

a)
$$r_1 = y_2 - \frac{1}{(p+b)(p+a)}u$$

b) Both parameters a and b are included in the expression for r_1. Assume an additive uncertainty Δa in parameter a. The residual generator can then be written as

$$r_1 = y_2 - \frac{1}{(p+b)(p+a+\Delta)}u = y_2 - \frac{1 - \frac{\Delta a}{p+a+\Delta a}}{(p+b)(p+a)}u$$

whose internal form becomes

$$r_1 = -\frac{\Delta a}{p+a+\Delta a}\frac{1}{(p+b)(p+a)}u$$

in the fault free case. The effect of the uncertainty in parameter b is treated equivalently.

c)
$$r_2 = \frac{1}{p+\alpha}(y_1 - (p+b)y_2)$$

d) r_2

Exercise 4.15.

a) If $z = \begin{bmatrix} y \\ u \end{bmatrix}$ the model is written:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(p+a) & 0 & 0 \\ 0 & -p & 1 \\ 0 & -ab & -(p+a+b) \end{bmatrix} w + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f = 0$$

b)
$$N_H(p) = \begin{bmatrix} p+a & 0 & 1 & 0 & 0 \\ 0 & (p+a)(p+b) & 0 & p+a+b & 1 \end{bmatrix}$$
$$N_H(p)L(p)z = \begin{bmatrix} -(p+a) & 0 & 1 \\ 0 & -(p+a)(p+b) & 1 \end{bmatrix} z = \begin{pmatrix} -(p+a)y_1 + u \\ -(p+a)(p+b)y_2 + u \end{pmatrix}$$

Exercise 4.16.

a)

$$\begin{bmatrix} 1+p & 0 & -1 \\ -1 & p & 0 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = 0$$

b)

$$\begin{aligned} r_1 &= \frac{1}{1+sT_d}[0.58 + 0.58s \;\; 0 \quad -0.58] \begin{pmatrix} y \\ u \end{pmatrix} \\ r_2 &= \frac{1}{1+sT_d}[-1 \;\; s \;\; 0] \begin{pmatrix} y \\ u \end{pmatrix} \end{aligned}$$

Exercise 4.18.

a) The controller does not affect

b) The controller affects

Exercise 4.19.

$$\begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f$$

There are two possible consitency relations that can be used to detect f,

$$\begin{aligned} y_2 s(s+1) - u &= s(s+1)f \\ y_2 s - y_1 &= sf \end{aligned}$$

In both versions only sf i.e. $\dot{f}$ is included, therefore the fault cannot be strongly detectable.

Exercise 4.20.

b) That the fault is detectable is follows directly from Rank $H(s) <$ Rank $[H(s)\; F(s)]$.

d) Calculate $N_H(s)F(s)$ from which one sees that the fault is not strongly detectable.

Exercise 4.21.

a)

$$M(s) = \begin{bmatrix} \frac{s+1}{s(s+2)} \\ 1 \end{bmatrix}$$

which yields

$$N_M(s) = [s(s+2) \quad -(s+1)]$$

Evaulation of the following yields that f_s is not strongly detectable.

$$N_M(s) \begin{bmatrix} L(s) \\ 0 \end{bmatrix} |_{s=0} = [s(s+2) \quad -(s+1)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} |_{s=0} = s(s+2)|_{s=0} = 0$$

b) In this case $M(s)$ is given by

$$M(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After some thinking:

$$N_M(s) = [s(s+1) \quad -(s+1) \quad -s]$$

Multiplication as in task a:

$$N_M(s) \begin{bmatrix} L(s) \\ 0 \end{bmatrix} |_{s=0} = [s(s+1) \quad -(s+1) \quad -s] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} |_{s=0} = s(s+21|_{s=0} = 0$$

Exercise 4.22.

b) Verify for example that $P = [2 \ \ -1]$, $A_z = -1$, $L_1 = [2 \ \ -1]$, $L_2 = 1$, $K = 2$ satisfies the conditions in task a.

Answers for Chapter 5

Nonlinear Residual Generation

Exercise 5.1. A consistency relation is given by

$$0 = \dot{y} - 2x\dot{x} = \dot{y} + 2x^2u = \dot{y} + 2yu$$

Exercise 5.2.

a) Choose an observer and residual generator according to

$$\begin{aligned}\dot{\hat{x}} &= -\hat{x}u + K(y - \hat{x})\\ r &= y - \hat{x}\end{aligned}$$

Let the estimation error be $e = x - \hat{x}$. In the fault free case the following is true

$$\dot{e} = -e(u + K)$$

By choosing K as for instance $K = -u+\alpha$ where α is an arbitrary positive constant, we get a guaranteed stable residual generator.

b) With a additive fault in the sensor, the internal form becomes

$$\begin{aligned}\dot{e} &= -e(u + K) - Kf = -\alpha e + (u - \alpha)f\\ r &= e + f\end{aligned}$$

c) $u = 0$ yields a transfer from a constant fault to residual 0 and the fault is not detectable.

Exercise 5.3. This solution is rather long and might require prior knowledge in Lyapunov theory, knowledge that is outside the scope of a course in diagnosis.

In that case, view this solution as a demonstration of how a proof of stability could look.

For the observer

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + K_1(y - \hat{x}_1)\\ \dot{\hat{x}}_2 &= -\hat{x}_2 - \sin\hat{x}_1 + u + K_2(y - \hat{x}_1)\end{aligned}$$

the error dynamics, with $e_i = x_i - \hat{x}_i$, can be written as

$$\begin{aligned}\dot{e}_1 &= e_2 - K_1 e_1\\ \dot{e}_2 &= -e_2 - K_2 e_1 + \sin\hat{x}_1 - \sin x_1\end{aligned}$$

Now take the suggested Lyapunov function $V(e) = \frac{1}{2}e_1^2 + \frac{1}{2}\beta e_2^2$ with $\beta > 0$. Then the following holds

$$\begin{aligned}\dot{V} = e_1\dot{e}_1 + \beta e_2\dot{e}_2 &= e_1(e_2 - K_1 e_1) + \beta e_2(-e_2 - K_2 e_1 + \sin\hat{x}_1 - \sin x_1) =\\ &= -K_1 e_1^2 - \beta e_2^2 + e_1 e_2(1 - \beta K_2 - \beta \underbrace{\frac{\sin x_1 - \sin\hat{x}_1}{x_1 - \hat{x}_1}}_{\gamma})\end{aligned}$$

According to the hint in the task $0 \le \gamma \le 1$. Now rewrite the quadratic form in matrix form

$$\dot{V} = \begin{pmatrix} e_1 & e_2 \end{pmatrix}\begin{pmatrix} -K_1 & \frac{1}{2}(1 - \beta K_2 - \beta\gamma) \\ \frac{1}{2}(1 - \beta K_2 - \beta\gamma) & -\beta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = e^T Q e$$

If we can select K so that Q is a negative definite matrix we're done. Basic linear algebra can be used to find conditions through criteria on the matrix minors. For instance $Q < 0$ is equivalent to

$$\begin{aligned}-K_1 &< 0\\ K_1\beta - \frac{1}{4}(1 - \beta K_2 - \beta\gamma)^2 &> 0\end{aligned}$$

Since we know that $0 \le \gamma \le 1$ this can be written as

$$\begin{aligned}0 <& K_1\\ -2\sqrt{\frac{K_1}{\beta}} + \frac{1-\beta}{\beta} <& K_2 < 2\sqrt{\frac{K_1}{\beta}} + \frac{1}{\beta}\end{aligned}$$

If a triple can be found, that fulfills the above conditions, $\dot{V} < 0$ and therefore the observer will be stable,

An example of a triple that fulfills the conditions is$< K_1, K_2, \beta > = < 1, 1, 1 >$, i.e. the observer

$$\dot{\hat{x}} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \sin\hat{x}_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(y - \hat{x}_1)$$

Exercise 5.5.

a) Select simple additive fault models for the sensors and the pump, yielding

$$
\begin{aligned}
\dot{h}_1 &= a_1(u + f_u) - a_2\sqrt{h_1} \\
\dot{h}_2 &= a_2\sqrt{h_1} - a_3\sqrt{h_2} \\
y_1 &= h_1 + f_1 \\
y_2 &= h_2 + f_2 \\
y_3 &= a_4\sqrt{h_1} + f_3 \\
y_4 &= a_5\sqrt{h_2} + f_4
\end{aligned}
$$

where $f_i = 0$ corresponds to a fault free component and $f_i \neq 0$ a with component fault.

b) For instance a changed flow from the upper tank gives the model

$$
\begin{aligned}
\dot{h}_1 &= a_1(u + f_u) - (1 - f_c)a_2\sqrt{h_1} \\
\dot{h}_2 &= (1 - f_c)a_2\sqrt{h_1} - a_3\sqrt{h_2} \\
y_1 &= h_1 + f_1 \\
y_2 &= h_2 + f_2 \\
y_3 &= (1 - f_c)a_4\sqrt{h_1} + f_3 \\
y_4 &= a_5\sqrt{h_2} + f_4
\end{aligned}
$$

where $f_c = 0$ is fault free and $f_c > 0$ clogged.

c) Exist relatively many potential consistency relations, one example is

$$y_4 - a_5\sqrt{y_2} = 0,$$

$$\dot{y}_2 - a_2\sqrt{y_1} + a_3\sqrt{y_2} = 0,$$

and

$$\dot{y}_4 = a_5\frac{1}{2\sqrt{h_2}}(a_2\sqrt{h_1} - a_3\sqrt{h_2}) = \frac{a_5^2 a_2}{2a_4}\frac{a_4\sqrt{h_1}}{a_5\sqrt{h_2}} - \frac{a_3 a_5}{2} = \frac{a_5^2 a_2}{2a_4}\frac{y_3}{y_4} - \frac{a_3 a_5}{2}$$

Exercise 5.6.

a) An example of a consistency relation with the sought after fault sensitivity is

$$\dot{y}_1 + a_2\sqrt{y_1} - a_1 u = 0$$

Since the derivatives enter linearly, we can use the same techniques as for linear systems, i.e.

$$\beta r + \dot{r} = \dot{y}_1 + a_2\sqrt{y_1} - a_1 u, \quad \beta > 0$$

which with the state$w = r - y_1$ can be written on state-space form

$$\begin{aligned}\dot{w} &= -\beta w - \beta y_1 + a_2\sqrt{y_1} - a_1 u\\ r &= w + y_1\end{aligned}$$

b) A consistency relation with the sought after fault sensitivity is

$$2y_3\dot{y}_3 - a_4^2 a_1 u + a_2 a_4 y_3$$

received through differentiation of the measurement signal y_3.

The y_3 before $\dot{y}_3$ is problematic, the derivative does not enter linearly. If we would have been able to divide both sides with y_3 the situation would have been the same as in task a, i.e. the derivative would have acted linearly, and we could have added linear dynamics and used the same methodology here. Now this is not possible and we need to use a different scheme. Note that

$$2y_3\dot{y}_3 = \frac{d}{dt}y_3^2$$

which gives that the consistency relation with added first order linear residual generator dynamics can be written

$$\beta r + \dot{r} = \frac{d}{dt}y_3^2 - a_4^2 a_1 u + a_2 a_4 y_3, \quad \beta > 0$$

which can be realized in state-space form with the state $w = r - y_3^2$ according to

$$\begin{aligned}\dot{w} &= -\beta w - \beta y_3^2 - a_4^2 a_1 u + a_2 a_4 y_3\\ r &= w + y_3^2\end{aligned}$$

Exercise 5.7. Assume additive fault models for all faults.

a) With the given fault sensitivity, the following equations can be used

$$\begin{aligned}\dot{h}_2 &= a_2\sqrt{h_1} - a_3\sqrt{h_2}\\ y_1 &= h_1\\ y_2 &= h_2\end{aligned}$$

and a residual generator can for instance be constructed according to

$$\begin{aligned}\dot{\hat{h}}_2 &= a_2\sqrt{y_1} - a_3\sqrt{\hat{h}_2} + K(y_2 - \hat{h}_2)\\ r &= y_2 - \hat{h}_2\end{aligned}$$

b)

$$\begin{aligned}\dot{\hat{h}}_2 &= \frac{a_2}{a_4}y_3 - a_3\sqrt{\hat{h}_2} + K(y_2 - \hat{h}_2)\\ r &= y_2 - \hat{h}_2\end{aligned}$$

c) With the given fault sensitivity, only the following equations

$$\begin{aligned}\dot{h}_2 &= a_2\sqrt{h_1} - a_3\sqrt{h_2}\\ y_2 &= h_2\end{aligned}$$

are left. This cannot be used to construct a residual. Can be seen since there are two equations and two unknowns, there is therefore no redundancy.

d) Using the constant error assumption the following equations are at our disposal

$$\begin{aligned}\dot{f}_u &= 0\\ \dot{h}_1 &= a_1(u + f_u) - a_2\sqrt{h_1}\\ \dot{h}_2 &= a_2\sqrt{h_1} - a_3\sqrt{h_2}\\ y_2 &= h_2\end{aligned}$$

An observer based residual generator are for example given by

$$\begin{aligned}\dot{\hat{f}}_u &= 0 + K_1(y_2 - \hat{h}_2)\\ \dot{\hat{h}}_1 &= a_1(u + \hat{f}_u) - a_2\sqrt{\hat{h}_1} + K_2(y_2 - \hat{h}_2)\\ \dot{\hat{h}}_2 &= a_2\sqrt{\hat{h}_1} - a_3\sqrt{\hat{h}_2} + K_3(y_2 - \hat{h}_2)\\ r &= y_2 - \hat{h}_2\end{aligned}$$

e) **Observers:**
Observers are often straight forward to formulate, however, it can often be tricky to compute a stabilizing feedback. Decoupling of faults, except sensor faults, can be harder to accomplish than in the consistency relation case.

Consistency Relations:
Elimination of variables in nonlinear equation systems, has more than exponential computational complexity in the number of variables. Thus the equation systems have to be of relatively small size, or to the largest part linear, for the elimination problem to be solvable. The resulting consistency relation can become quite complex and long to write, even for relatively small equations systems(ca 10 equations). Since dynamic equations are included in the equation system the consistency relation will include derivatives and potentially higher order derivatives of known signals, something that has to be handled e.g. through estimation straight from measurements or, when possible, through low pass filtering as in the

linear case. One advantage using consistency relations is the easy manner in which decoupling is dealt with.

Summary: In decoupling of faults (not sensor faults) consistency relations can be advantageous. If the model, on which the tests are based, contain many unknowns or a lot of dynamics (corresponding to higher order derivatives of known variables in the consistency relations) it can be more advantageous to use observer methodology.

Exercise 5.8.

a)

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{f}}_1 \\ \dot{\hat{f}}_2 \end{pmatrix} = \begin{pmatrix} g_1(\hat{x}_1, \hat{f}_1, u) \\ g_2(\hat{x}_1, \hat{x}_2) \\ 0 \\ 0 \end{pmatrix} + K \begin{pmatrix} y_1 - h_1(\hat{x}_1) \\ y_2 - h_2(\hat{x}_2) - \hat{f}_2 \end{pmatrix}$$

b)

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{f}}_1 \end{pmatrix} = \begin{pmatrix} g_1(\hat{x}_1, \hat{f}_1, u) \\ g_2(\hat{x}_1, \hat{x}_2) \\ 0 \end{pmatrix} + K \begin{pmatrix} y_1 - h_1(\hat{x}_1) \\ y_2 - h_2(\hat{x}_2) \end{pmatrix}$$

och en residual $r = |y - \hat{y}|$.

c) Here we can use just the upper part of the system and construct an observer

$$\dot{\hat{x}}_1 = g_1(\hat{x}_1, 0, u) + K(y_1 - h_1(\hat{x}_1))$$

with a residual $r = y_1 - h_1(\hat{x}_1)$. One could do the same as in the answer to task b, but it is not necessary and is also independent of that the model for x_2 is correct.

d) As seen above, only the constant fault assumption on f_1 is required for the simple methods outlined to be directly applicable.

Note however, that it is possible, but probably more complicated, depending on the shape of the nonlinear functions g_i and h_i, to construct an observer solution even without the constant fault assumption on f_1.

Exercise 5.9.

$$\dot{y}^2 - u\dot{y} - \ddot{y}y + \dot{u}y = 0$$

Exercise 5.10.

⋆c)

$$h_1 = 27(x^3 + y)(y^2 - (u - x^2)^3)$$
$$h_2 = 9x^4(u - x^2)^2 + 27uy^2 + 3x^2(u - x^2)\dot{y} + \dot{y}^2$$

Exercise 5.11.

a) Assign

$$\tilde{Y} = \begin{bmatrix} y(t) - u_1(t) \\ y(t-1) - u_1(t-1) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ u_2(t-1) \end{bmatrix} \theta_2$$

Least squares estimation of θ_2 becomes

$$\hat{\theta}_2 = (U^T U)^{-1} U^T \tilde{Y} =$$
$$= \frac{(u_2(t)(y(t) - u_1(t)) + u_2(t-1)(y(t-1) - u_1(t-1)))}{u_2^2(t) + u_2^2(t-1)}$$

A residual can now be computed according to

$$r_1(t) = y(t) - u_1(t) - \hat{\theta}_2 u_2(t) = \cdots =$$
$$= \frac{u_2(t-1)}{u_2^2(t) + u_2^2(t-1)} (y(t)u_2(t-1) - u_1(t)u_2(t-1) + u_1(t-1)u_2(t) - y(t-1)u_2(t))$$

b)

$$r_2(t) = y(t)u_2(t-1) - u_1(t)u_2(t-1) + u_1(t-1)u_2(t) - y(t-1)u_2(t)$$

c)

$$r_1(t) = \frac{u_2(t-1)}{u_2^2(t) + u_2^2(t-1)} r_2(t)$$

Answers for Chapter 6

Multiple fault isolation

Exercise 6.1. The set of minimal diagnoses for each observation are {"no faults"}, {"S stuck closed"}, {"S stuck open", "L broken"} and {"no faults"}.

Exercise 6.2.

a) The sets in Definition 2.1 will be for this example:

$$\begin{aligned}\mathcal{M} = \{&OK(M1) \rightarrow x = ac, &(6.1)\\ &OK(M2) \rightarrow y = bd, &(6.2)\\ &OK(M3) \rightarrow z = ce, &(6.3)\\ &OK(A1) \rightarrow f = x + y, &(6.4)\\ &OK(A2) \rightarrow g = y + z\} &(6.5)\end{aligned}$$

$$obs = \{a = 3, b = 3, c = 2, d = 1, e = 5, f = 9, g = 12\} \qquad (6.6)$$

$$\mathcal{D} = \{OK(M1) \wedge OK(M2) \wedge OK(M3) \wedge OK(A1) \wedge \neg OK(A2)\} \qquad (6.7)$$

The assigned modes in (6.7) result in that the equations (6.1)-(6.4) are consistent. By assigning the variables as (6.6) we get

$$\begin{aligned}3 \cdot 2 &= x\\ 3 \cdot 1 &= y\\ 2 \cdot 5 &= z\\ x + y &= 9\end{aligned}$$

The equations are consistent since there exists a solution $x = 6$, $y = 3$, and $z = 10$. According to Definition 2.1, qthe allocated modes in (6.7) is a diagnosis.

b) One example of a conflict is:

$$OK(M2) \land OK(M3) \land OK(A2)$$

This is a conflict since all components $M2$, $M3$, and $A2$ cannot be OK, since g should then be 13.

With similar reasoning, you can also see that

$$OK(M1) \land OK(M3) \land OK(A1) \land OK(A2)$$

is a conflict.

c) The minimal conflicts are:

$$OK(M2) \land OK(M3) \land OK(A2)$$
$$OK(M1) \land OK(M3) \land OK(A1) \land OK(A2)$$

Exercise 6.3. We use the notation A and $\neg A$ to represent $OK(A)$ and $\neg OK(A)$, respectively, to simplify the notations.

a) The negated conflicts are

$$\neg\pi_1 = \neg A \lor \neg B, \quad \neg\pi_2 = \neg B \lor \neg C$$

For the diagnosis $\mathcal{D}_1$ it comes to that the set of expressions $\{\neg\pi_1, \neg\pi_2\} \cup \mathcal{D}_1$ is satisfiable is equivalent to that the following expression is satisfiable

$$(\neg A \lor \neg B) \land (\neg B \lor \neg C) \land A \land \neg B \land C = A \land \neg B \land C$$

which it is. Tips: $X \land (X \lor Y) = X$.

In the same way it can be shown that $\mathcal{D}_2$ is a diagnosis but for the set $\{\neg\pi_1, \neg\pi_2\} \cup \mathcal{D}_3$ we get the expression

$$(\neg A \lor \neg B) \land (\neg B \lor \neg C) \land A \land B \land \neg C = (\neg A \lor \neg B) \land A \land B \land \neg C = \text{falskt}$$

which is not satisfiable. Tips: $X \land (\neg X \lor Y) = X \land Y$.

b) Using set notation the two conflicts becomes

$$\pi_1 = \{A, B\}, \quad \pi_2 = \{B, C\}$$

and the three mode allocations

$$\mathcal{D}_1 = \{\neg B\}$$
$$\mathcal{D}_2 = \{\neg A, \neg C\}$$
$$\mathcal{D}_3 = \{\neg C\}$$

with the convention that the set of mode allocations only considers faulty components. Direct use of Theorem 3.6 give that $\mathcal{D}_1$ and $\mathcal{D}_2$ are diagnoses while $\mathcal{D}_3$ is not.

c) The only single-fault diagnosis is

$$\neg\pi_1 \cap \neg\pi_2 = \{\neg A, \neg B\} \cap \{\neg B, \neg C\} = \{\neg B\}$$

Exercise 6.4.

a) If no residual has triggered, there is no conflict. If r_2 has tirggered, then $OK(A) \wedge OK(C)$ en konfliktis a conflict.

b) The intersection of the decisions result in an empty set of diagnoses. The conflicts are

$$\begin{aligned}&OK(B) \wedge OK(C)\\&OK(A) \wedge OK(C)\\&OK(B)\end{aligned}$$

The minimal diagnoses are

$$\begin{aligned}\neg OK(A) \wedge \neg OK(B) \wedge OK(C)\\OK(A) \wedge \neg OK(B) \wedge \neg OK(C)\end{aligned}$$

The first method do not handle multiple-faults, except if the decision structure is extended with additional columns corresponding to the multiple-faults.

Exercise 6.5.

a)

$$\begin{aligned}\pi_1 &= OK(A) \wedge OK(B)\\\pi_2 &= OK(C) \wedge OK(D)\\\pi_3 &= OK(A) \wedge OK(D)\\\pi_4 &= OK(A) \wedge OK(C)\\\pi_5 &= OK(B) \wedge OK(D)\end{aligned}$$

b) The minimal diagnoses are

$$\begin{aligned}\mathcal{D}_1 &= \neg OK(A) \wedge OK(B) \wedge OK(C) \wedge \neg OK(D)\\\mathcal{D}_2 &= \neg OK(A) \wedge \neg OK(B) \wedge \neg OK(C) \wedge OK(D)\\\mathcal{D}_3 &= OK(A) \wedge \neg OK(B) \wedge \neg OK(C) \wedge \neg OK(D)\end{aligned}$$

Exercise 6.6.

a) $4^{30} \approx 1.15 \cdot 10^{18}$

b) $|S_1^1| = 4^{29} \cdot 3 \approx 8.65 \cdot 10^{17}$

c) $1 + 3 \cdot 30 + \frac{30!}{2!28!} \cdot 9 = 4006$

d) $|S_1^1| = 3 + 29 \cdot 9 = 264$

Exercise 6.7.

a) Let OK denote the fault-free mode, $\neg OK$ broken, SC stuck closed, and SO stuck open. The switch's requested position can be open O or closed C and the lamp and the LED can be lit L or not $\neg L$. Then, the minimal conflicts for each observation are:

Observation			Minimal conflicts
D	L	S	
$\neg L$	$\neg L$	$O \vee C$	$\{OK(B)\}$
$\neg L$	L	$O \vee C$	$\emptyset$
L	$\neg L$	O	$\{\neg OK(B)\}$, $\{SC(S), OK(L)\}$
L	$\neg L$	C	$\{\neg OK(B)\}$, $\{OK(S), OK(L)\}$, $\{SC(S), OK(L)\}$
L	L	O	$\{\neg OK(B)\}$, $\{OK(S)\}$, $\{SO(S)\}$, $\{\neg OK(L)\}$
L	L	C	$\{\neg OK(B)\}$, $\{SO(S)\}$, $\{\neg OK(L)\}$

b) The minimal diagnoses are underlined.

Observation			Diagnosis
D	L	S	
$\neg L$	$\neg L$	$O \vee C$	$\underline{\{\neg OK(B)\}}$, $\{\neg OK(B), SO(S)\}$, $\{\neg OK(B), SC(S)\}$, $\{\neg OK(B), \neg OK(L)\}$, $\{\neg OK(B), \neg OK(L), SO(S)\}$, $\{\neg OK(B), \neg OK(L), SC(S)\}$
$\neg L$	L	$O \vee C$	no diagnoses
L	$\neg L$	O	$\underline{\emptyset}$, $\{SO(S)\}$, $\{\neg OK(L)\}$, $\{\neg OK(L), SO(S)\}$, $\{\neg OK(L), SC(S)\}$
L	$\neg L$	C	$\underline{\{SO(S)\}}$, $\underline{\{\neg OK(L)\}}$, $\{\neg OK(L), SO(S)\}$, $\{\neg OK(L), SC(S)\}$
L	L	O	$\underline{\{SC(S)\}}$
L	L	C	$\underline{\emptyset}$, $\{SC(S)\}$

As an example of how the diagnoses can be computed from the conflicts consider the case on the third row. Using logic notation, the negated conflicts are

$$OK(B) \wedge (\neg SC(S) \vee \neg OK(L))$$

Since $S \in \{OK, SO, SC\}$, $\neg SC(S) \equiv OK(S) \vee SO(S)$. By using this in the expression above and expanding the expression to a conjunction of disjunctions we get

$$\begin{aligned} OK(B) \wedge (OK(S) \vee SO(S) \vee \neg OK(L)) &\equiv \\ (OK(B) \wedge OK(S)) \wedge (OK(B) \vee SO(S)) \wedge (OK(B) \vee \neg OK(L)) \end{aligned}$$

All mode allocations implying any of the conjunctions above are diagnoses. For example, $OK(B) \wedge OK(S)$ is implied by $OK(B) \wedge OK(S) \wedge OK(L)$ but also $OK(B) \wedge OK(S) \wedge \neg OK(L)$ which are two of the diagnoses.

c) Yes.

d) No.

e) The kernel diagnoses for each observation are:

Observation			Kernel diagnoses
D	L	S	
$\neg L$	$\neg L$	$O \vee C$	$\neg OK(B)$
$\neg L$	L	$O \vee C$	no diagnoses
L	$\neg L$	O	$OK(B) \wedge OK(S),\ OK(B) \wedge SO(S),\ OK(B) \wedge \neg OK(L)$
L	$\neg L$	C	$OK(B) \wedge SO(S)$, $OK(B) \wedge \neg OK(L)$
L	L	O	$OK(B) \wedge SC(S) \wedge OK(L)$
L	L	C	$OK(B) \wedge OK(S) \wedge OK(L),\ OK(B) \wedge SC(S) \wedge OK(L)$

Exercise 6.8.

a) Introduce the interpretation T that is true if the lamp is lit. The model is

$$\begin{aligned} \mathcal{M} = \{ & OK(B) \to E & (6.8)\\ & OK(L_1) \to (E \leftrightarrow T(L_1)) & (6.9)\\ & OK(L_2) \to (E \leftrightarrow T(L_2))\} & (6.10) \end{aligned}$$

and the observation

$$\begin{aligned} obs = \{ & \neg T(L_1), & (6.11)\\ & T(L_2)\} & (6.12) \end{aligned}$$

b) According to Definition 2.1, there must be a value of E such that $\mathcal{M} \bigcup obs \bigcup \mathcal{D}$ is satisfiable. We start with testing if $OK(B) \wedge OK(L_1)$ is consistent with the model and observations. From $OK(B)$ and (6.8) follows E. The expressions $OK(L_1)$ and (6.9) imply that $E \leftrightarrow T(L_1)$. This and (6.11) give that $\neg E$. This is a contradiction to E which means that $OK(B) \wedge OK(L_1)$ is not consistent with the model and the observations. Thus, neither $OK(B) \wedge OK(L_1) \wedge OK(L_2)$ nor $OK(B) \wedge OK(L_1) \wedge \neg OK(L_2)$ are diagnoses. In the same way, the consistency of the following mode allocations are tested:

$$\begin{aligned} & OK(B) \wedge \neg OK(L_1) \wedge OK(L_2) \Rightarrow E\\ & OK(B) \wedge \neg OK(L_1) \wedge \neg OK(L_2) \Rightarrow E\\ & \neg OK(B) \wedge OK(L_1) \wedge OK(L_2) \Rightarrow \neg E \wedge E \Leftrightarrow \bot\\ & \neg OK(B) \wedge OK(L_1) \wedge \neg OK(L_2) \Rightarrow \neg E\\ & \neg OK(B) \wedge \neg OK(L_1) \wedge OK(L_2) \Rightarrow E\\ & \neg OK(B) \wedge \neg OK(L_1) \wedge \neg OK(L_2)\} \Rightarrow E \vee \neg E \end{aligned}$$

This means that the diagnoses are $\{\neg OK(L_1)\}$, $\{\neg OK(B), \neg OK(L_2)\}$, $\{\neg OK(L_1), \neg OK(L_2)\}$, $\{\neg OK(B), \neg OK(L_1)\}$ och $\{\neg OK(B), \neg OK(L_1), \neg OK(L_2)\}$. The minimal diagnoses are $\{\neg OK(L_1)\}$ and $\{\neg OK(B), \neg OK(L_2)\}$.

c) The following is included in $\mathcal{M}$:

$$\begin{aligned} & \neg OK(B) \to \neg E\\ & \neg OK(L_1) \to \neg(E \leftrightarrow T(L_1))\\ & \neg OK(L_2) \to \neg(E \leftrightarrow T(L_2)) \end{aligned}$$

The model can now be described as:

$$OK(B) \leftrightarrow E$$
$$OK(L_1) \leftrightarrow (E \leftrightarrow T(L_1))$$
$$OK(L_2) \leftrightarrow (E \leftrightarrow T(L_2))$$

The diagnoses are $\{\neg OK(B), \neg OK(L_2)\}$ and $\{\neg OK(L_1)\}$

d) Fault models must be included in the model.

e) Possible fault modes are:

$$\neg OK(B) \rightarrow \neg E$$
$$\neg OK(L_1) \rightarrow \neg T(L_1)$$
$$\neg OK(L_2) \rightarrow \neg T(L_2)$$

The only remaining diagnosis is $\{\neg OK(L_1)\}$.

Exercise 6.9.

a) The conflicts are

$$OK(A) \wedge OK(B)$$
$$SA0(A) \wedge OK(B)$$
$$SHORT(A) \wedge SHORT(B)$$

and the diagnoses are

$$OK(A) \wedge SA0(B)$$
$$OK(A) \wedge SHORT(B)$$
$$SA0(A) \wedge SA0(B)$$
$$SA0(A) \wedge SHORT(B)$$
$$SHORT(A) \wedge OK(B)$$
$$SHORT(A) \wedge SA0(B)$$

b) The new conflicts are

$$SA0(B)$$
$$SA0(A) \wedge SHORT(B)$$

and the diagnoses are

$$OK(A) \wedge SHORT(B)$$
$$SHORT(A) \wedge OK(B)$$

Exercise 6.12.

a) The decision structure is

	NF	F_1	F_2	F_3
δ_1	0	X	0	0
δ_2	0	0	X	0
δ_3	0	0	0	X

b) The decisions structure is

	NF	F_1	F_2	F_3	F_{12}	F_{13}	F_{23}
δ_1	0	X	0	0	X	X	0
δ_2	0	0	X	0	X	0	X
δ_3	0	0	0	X	0	X	X

c) The three double-faults can be uniquely isolated.

d) Given that the principle to determine the sensitivity to double-faults of the different tests in b is a model property it will be impossible to isolate single-faults.

Exercise 6.13.

a) Let all observations consistent with mode $A1$ and $A1\&M1$ be denoted as $O(A1)$ and $O(A1\&M1)$, respectively. These sets are given by

$$O(A1) = \{(a, b, c, d, e, f, g) | \exists f_f \neq 0 . ac + bd + f_f = f\}$$
$$O(A1\&M1) = \{(a, b, c, d, e, f, g) | \exists \hat{f}_x \neq 0 \exists \hat{f}_f \neq 0 . (ac + \hat{f}_x) + bd + \hat{f}_f = f\}$$

To show that $A1$ is not isolable from $A1\&M1$ is equivalent to show that $O(A1) \subseteq O(A1\&M1)$. Assume that z is the vector with known variables. We will show that for any observation $z \in O(A1)$ it holds that $z \in O(A1\&M1)$. Select any $z_0 \in O(A1)$ which is equivalent to that there is a $f_f^0 \neq 0$ such that

$$a_0 c_0 + b_0 d_0 + f_f^0 = f_0 \tag{6.13}$$

To determine if $z_0 \in O(A1\&M1)$ use the value of z_0 in the expression for $A1\&M1$, i.e.,

$$a_0 c_0 + b_0 d_0 + \hat{f}_x + \hat{f}_f = f_0 \tag{6.14}$$

Eliminating a_0, b_0, c_0, d_0, and f_0 in equations (6.13) and (6.14) gives $O(A1\&M1)$ there should be an $\hat{f}_x \neq 0$ and an $\hat{f}_f \neq 0$ such that $f_f^0 = \hat{f}_x + \hat{f}_f$. Since $f_f^0 \neq 0$ is such a choice $\hat{f}_x = \hat{f}_f = f_f^0/2$, which proves the statement.

b) Assume that the fault in $A1\&M1$ is $\hat{f}_x = -\hat{f}_f \neq 0$. According to the relation in a, it will give the same observations as when $f_f = -\hat{f}_f + \hat{f}_f = 0$ showing that $A1$ cannot explain the fault.

c) The mode $A2\&M1$ gives that

$$\begin{aligned} ac + f_x + bd &= f \\ bd + ce + f_g &= g \end{aligned}$$

where $f_x \neq 0$ and $f_g \neq 0$. Mode $M2\&M3$ gives that

$$\begin{aligned} ac + bd + f_y &= f \\ bd + f_y + ce + f_z &= g \end{aligned}$$

Elimination of known variables from the two systems gives that

$$\begin{aligned} f_x &= f_y \\ f_g &= f_y + f_z \end{aligned} \tag{6.15}$$

If $f_x = f_g \neq 0$ in $A2\&M1$ then $f_z = 0$ to satisfy (6.15). Thus, there are faults in $A2\&M1$ that cannot be explained by $M2\&M3$. The converse is also true, since if $f_y = -f_z$ then $f_g = 0$ according to (6.15).

d) The sensitivity to faults in each residual becomes:

$$\begin{aligned} T_0 &= |f_x + f_y + f_f| + |f_y + f_z + f_g| \\ T_1 &= |f_y + f_z + f_g| \\ T_2 &= |f_x + f_f - f_z - f_g| \\ T_3 &= |f_x + f_y + f_f| \end{aligned}$$

The decision structure is determined by the following rules. In position (T_i, B), where T_i is a test quantity and B is a mode there should be a

0:a if T_i do not contain a non-zero variable in B.

1:a if T_i contain one absolute value with only one of the non-zero variables in T_i.

X otherwise.

The decision structure becomes:

	NF	A1	A2	M1	M2	M3	A1&A2	A1&M1
T_0	0	1	1	1	1	1	1	X
T_1	0	0	1	0	1	1	1	0
T_2	0	1	1	1	0	1	X	X
T_3	0	1	0	1	1	0	1	X

	A1&M2	A1&M3	A2&M1	A2&M2	A2&M3	M1&M2	M1&M3	M2&M3
T_0	1	1	1	1	X	1	1	1
T_1	1	1	1	X	X	1	1	X
T_2	1	X	X	1	X	1	X	1
T_3	X	1	1	1	0	X	1	1

e) The fault signature for $A1$ is $[1\,0\,1\,1]'$ which would lead to that $A1\&M1$ is an explanation. The column corresponding to $A1\&M1$ is $[X\,0\,X\,X]'$ meaning that there are many signatures for $A1\&M1$. Since the first row contains an X there must be a signature with 0 in the first row. This signature results in that $A1$ is not an explanation.

The analysis for $A2\&M1$ och $M2\&M3$ is done in the same way.

f) $S = \{A2, M3, A1\&M2, A2\&M3, M1\&M2\}$

g) $S = \{A2, M3, A2\&M3\}$

Answers for Chapter 7

Probabilistic Diagnosis

Exercise 7.1.

a) Let the binary stochastic variables H and T represent infected and test result respectively. Then we have the conditional probability tables

h	P(t|h)
falsk	1/10 000
sann	1

P(h)
1/10 000

The probabilities for the stochastic variable H are

$$P(H|t) = \alpha\langle\frac{1}{10\,000}(1-\frac{1}{10\,000}), 1\cdot\frac{1}{10\,000}\rangle \approx< 0.5, 0.5 >$$

b) The new probabilities are

$$P(H|t) = \alpha\langle\frac{1}{10\,000}(1-\frac{1}{100}), 1\cdot\frac{1}{100}\rangle \approx< 0.01, 0.99 >$$

Exercise 7.2.

a) The conditional probability tables are given by

f_1	$P(f_1)$
falsk	0.90
sann	0.10

f_1	$P(\text{alarm}\|f_1)$
falsk	0.05
sann	0.90

b) The case $F_1 = \text{sann}$ gives

$$P(f_1|\text{alarm}) = \alpha\, P(f_1, \text{alarm}) = \alpha\, P(\text{alarm}|f_1)P(f_1) = \alpha \cdot 0.9 \cdot 0.1$$

and the case $F_1 = \text{falsk}$ gives

$$P(\neg f_1|\text{alarm}) = \alpha\, P(\neg f_1, \text{alarm}) = \alpha\, P(\text{alarm}|\neg f_1)P(\neg f_1) = \alpha \cdot 0.05 \cdot 0.9$$

Normalization gives that

$$P(F_1|\text{alarm}) = \langle 0.33, 0.67\rangle$$

which corresponds to a normalization factor of

$$\alpha = \frac{1}{0.9 \cdot 0.1 + 0.05 \cdot 0.9} \approx 7.41$$

c) The conditional probability tables are

f_2	$P(f_2)$
sann	0.10
falsk	0.90

f_1	f_2	$P(\text{alarm}\|f_1, f_2)$
falsk	falsk	0.05
falsk	sann	0.60
sann	falsk	0.90
sann	sann	0.95

The basic equations for conditional probabilities and maginalization gives, for the case $F_1 = \text{true}$

$$P(f_1|\text{alarm}) = \alpha\, P(f_1, \text{alarm}) = \alpha \sum_{f_2} P(f_1, f_2, \text{alarm}) =$$

$$\alpha \sum_{f_2} P(\text{alarm}|f_1, f_2)P(f_1)P(f_2) = \alpha\,(P(\text{alarm}|f_1, f_2)P(f_1)P(f_2) +$$

$$+P(\text{alarm}|f_1, \neg f_2)P(f_1)P(\neg f_2)) = \alpha \cdot 0.0905$$

In the same way

$$P(\neg f_1|\text{alarm}) = \alpha \cdot 0.0945$$

and after normalization

$$P(F_1|\text{alarm}) = \langle 0.51, 0.49\rangle$$

The corresponding calculations given f_2 give

$$P(F_2|\text{alarm}) = \langle 0.66, 0.34\rangle$$

d) The probability of alarm is

$$P(\text{alarm}) = \sum_{f_1, f_2} P(\text{alarm}, f_1, f_2) = \sum_{f_1, f_2} P(\text{alarm}|f_1, f_2)P(f_1)P(f_2) \approx 0.1850$$

e) The stochastic variable FM can be either NF, f_1, f_2, or $f_1\&f_2$ and the probabilities for each fault mode are

$$\begin{aligned}
P(FM = NF|\text{alarm}) &= P(\neg f_1, \neg f_2|\text{alarm})\\
P(FM = f_1|\text{alarm}) &= P(f_1, \neg f_2|\text{alarm})\\
P(FM = f_2|\text{alarm}) &= P(\neg f_1, f_2|\text{alarm})\\
P(FM = f_1\&f_2|\text{alarm}) &= P(f_1, f_2|\text{alarm})
\end{aligned}$$

which corresponds to

$$P(FM|\text{alarm}) = \langle 0.2189, 0.4378, 0.2919, 0.0514\rangle$$

Exercise 7.3. The probabilities are

$$\begin{aligned}
P(B|j) &\approx \langle 0.984,\ 0.016\rangle \\
P(B|m) &\approx \langle 0.944,\ 0.056\rangle \\
P(E|j,m) &\approx \langle 0.824,\ 0.176\rangle \\
P(M|b) &\approx \langle 0.341,\ 0.659\rangle \\
P(J|\neg b) &\approx \langle 0.949,\ 0.051\rangle
\end{aligned}$$

Exercise 7.4.

a) Minimal diagnoses: f_3 och $f_1\&f_2$.

c) The probabilities of the three faults are

$$\begin{aligned}
P(F_1|t_1,t_2,\neg t_3) &= \langle 0.983, 0.017\rangle \\
P(F_2|t_1,t_2,\neg t_3) &= \langle 0.91, 0.09\rangle \\
P(F_3|t_1,t_2,\neg t_3) &= \langle 0.08, 0.92\rangle
\end{aligned}$$

The probability of the fault-free system, given the observed tests, can not be calculated as the product since even though the faults are independent, they are not independent conditioned the test results.

c)

e) The probabilities for the three faults when T_3 has not been evaluated is

$$\begin{aligned}
P(F_1|t_1,t_2) &= \langle 0.83, 0.17\rangle \\
P(F_2|t_1,t_2) &= \langle 0.76, 0.24\rangle \\
P(F_3|t_1,t_2) &= \langle 0.83, 0.17\rangle
\end{aligned}$$

The conclusion is that the faults f_1 and f_2 cannot be rejected since f_3 can explain that both T_1 and T_2 trigger. It is first when it is known that T_3 did not trigger, f_1 and f_2 can be rejected.

fault in F_3, reverse numbers?

Exercise 7.5.

a) The binary stochastic variable FAult is a deterministic *or*-function of the fault variables and

$$P(\text{Fault}|t_1,t_2,\neg t_3) \approx \langle 0.045, 0.955\rangle$$

b) The probabilities are

$$\begin{aligned}
P(F_1|t_1,t_2,\neg t_3,\text{fault}) &= \langle 0.983, 0.017\rangle \\
P(F_2|t_1,t_2,\neg t_3,\text{fault}) &= \langle 0.905, 0.095\rangle \\
P(F_3|t_1,t_2,\neg t_3,\text{fault}) &= \langle 0.037, 0.963\rangle
\end{aligned}$$

c) The probabilities are

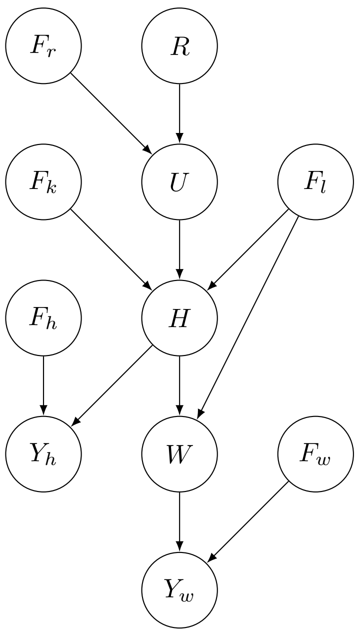
$$
\begin{aligned}
P(FM = NF|t_1, t_2, \neg t_3) &\approx \langle 0.045, 0.955\rangle \\
P(FM = F1|t_1, t_2, \neg t_3) &\approx \langle 0.996, 0.004\rangle \\
P(FM = F2|t_1, t_2, \neg t_3) &\approx \langle 0.971, 0.029\rangle \\
P(FM = F3|t_1, t_2, \neg t_3) &\approx \langle 0.15, 0.85\rangle \\
P(FM = F12|t_1, t_2, \neg t_3) &\approx \langle 0.999, 0.001\rangle \\
P(FM = F13|t_1, t_2, \neg t_3) &\approx \langle 0.989, 0.011\rangle \\
P(FM = F23|t_1, t_2, \neg t_3) &\approx \langle 0.941, 0.059\rangle \\
P(FM = F123|t_1, t_2, \neg t_3) &\approx \langle 0.000, 1.000\rangle
\end{aligned}
$$

Exercise 7.6. The probabilities for the three faults are

$$
\begin{aligned}
P(F_1|t_1, t_2, \neg t_3) &= \langle 0.983, 0.017\rangle \\
P(F_2|t_1, t_2, \neg t_3) &= \langle 0.912, 0.088\rangle \\
P(F_3|t_1, t_2, \neg t_3) &= \langle 0.076, 0.924\rangle
\end{aligned}
$$

The differences compared to task 7.4-c are hardly visible. The main reason for the difference is how multiple-faults, which are unlikely, are dealt with.

Exercise 7.7. There are many approaches to model this type of system, the following solution is one approach. The structure of the Bayesian Network could look like this.



The variables are pump reference signal R, pump outflow U, tank level H, tank outflow W, and sensors Y_h and Y_W. Faults in the pump, level sensor, flow sensor, clogging, and leakage are represented by the variables F_r, F_h, F_w, F_k, and F_l respectively. Given a simplified model of the system, the conditional probability tables could look as below. The probability tables without parents:

r	$P(r)$
high	0.50
low	0.50

f_r	$P(f_r)$
true	0.10
false	0.90

f_h	$P(f_h)$
true	0.05
false	0.95

f_k	$P(f_k)$
true	0.20
false	0.80

f_w	$P(f_w)$
true	0.05
false	0.95

f_l	$P(f_l)$
true	0.05
false	0.95

The system functionality could be modeled as, for example,

f_r	r	$P(U = \text{high}\|f_r, r)$
false	low	0.01
false	high	0.99
true	low	0.1
true	high	0.1

f_k	f_l	u	$P(H = \text{high}\|u, f_l)$
false	false	low	0.10
false	false	high	0.90
false	true	low	0.05
false	true	high	0.70
true	false	low	0.20
true	false	high	0.95
true	true	low	0.10
true	true	high	0.80

f_l	h	$P(W = \text{high}\|f_l, h)$
false	low	0
false	high	0.95
true	low	0
true	high	0.7

and for the two sensors

f_h	h	$P(Y_h = \text{high}\|f_h, h)$
false	low	0.05
false	high	0.95
true	low	0.50
true	high	0.50

f_w	w	$P(Y_w = \text{high}\|f_w, w)$
false	low	0.05
false	high	0.95
true	low	0.50
true	high	0.50

Answers for Chapter 8

Fault Effect and Fault Tree Analysis

Exercise 8.1.

a) Start av FMEA:

Processsteg	Felsätt	Felorsak	Feleffekt	Riskanalys			
				OCC	SEV	DET	RPN
Hitta läckaget	Fanns fler läckage	Dålig noggranhet	Punktering	4	3	2	24
		Små hål	Smygpunka	3	5	4	60
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

b) & c) Riskbedömning för FMEA:n i a):

Felnr	OCC	SEV	RPN	Riskbedömning b)	Riskbedömning c)	Skillnad
1	4	3	24	T	T	
2	3	5	60	IT	BT	Ja
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Exercise 8.2. Ett exempel på hur ett felträd kan se ut. Här följer en tabell med de införda beteckningarna som används i felträdet. Vissa beteckningar har index f som står för fram och b som betyder bak.

A: Cykeln är inte lämplig att använda.
B: Cykeln rullar inte som den ska.
C: Handbromsarna fungerar inte.
D: Belysningen uppfyller inte lagkraven.
E: Framhjulet fungerar inte.
F: Bakhjulet fungerar inte.
G: Frambroms fungerar ej.
H: Bakbroms fungerar ej.
I: Det är mörkt ute.
J: Någon lampa fungerar ej.
L: Framdäcket är tomt på luft.
M: Frambromsen går emot fälgen.
N: Bakdäcket är tomt på luft.
O: Bakbromsen går emot fälgen.
P: Bromsvajern fungerar inte.
Q: Broms utsliten.
R: Framlyset är trasigt.
S: Baklyset är trasigt.
T: Ventilen fungerar inte.
U: Slangen är trasig.
V: Bromsvajern har frusit fast.
X: Bromsvajern är av.
Y: Batteriet är slut.
Z: Kontaktfel.
Å: Glödlampan är trasig.
Ä: Fukt i bromsvajern.
Ö: Minusgrader.

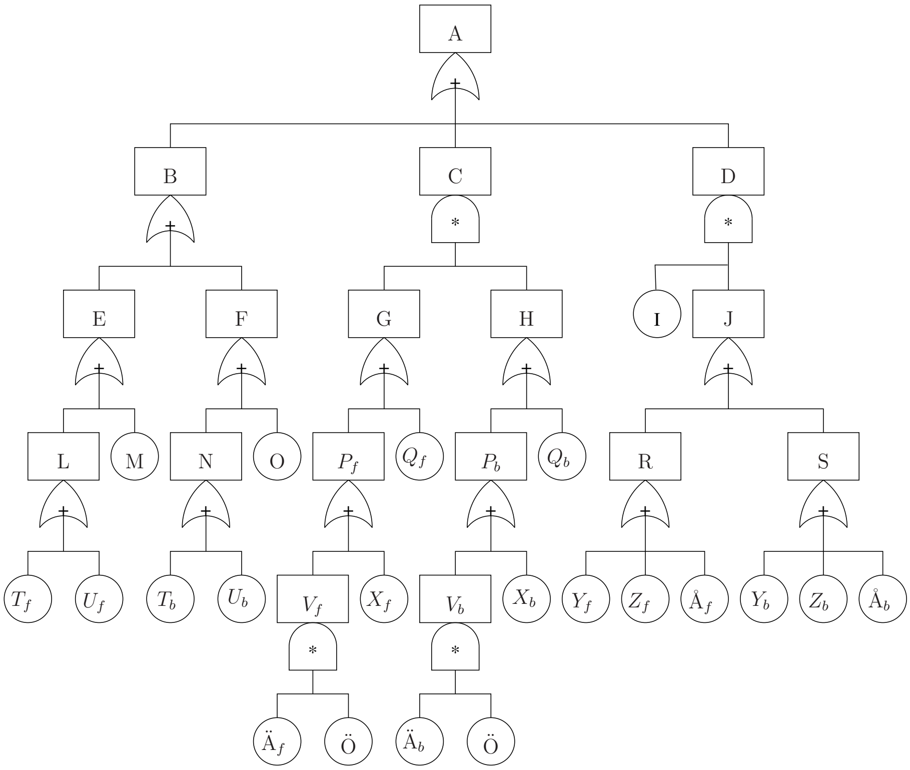


Figure 8.1: Exempel på ett felträd.

Exercise 8.3. $P(A) \approx 6$ ppm.

Exercise 8.4. De minimala avbrotten är $\{T_f\}$, $\{U_f\}$, $\{M\}$, $\{T_b\}$, $\{U_b\}$, $\{O\}$, $\{Ä_f, Ä_b, Ö\}$, $\{Ä_f, Ö, Q_b\}$, $\{Ä_f, Ö, X_b\}$, $\{Ä_b, Ö, Q_f\}$, $\{Ä_b, Ö, X_f\}$, $\{X_f, X_b\}$, $\{X_f, Q_b\}$, $\{Q_f, X_b\}$, $\{Q_f, Q_b\}$, $\{I, Y_f\}$, $\{I, Z_f\}$, $\{I, Å_f\}$, $\{I, Y_b\}$, $\{I, Z_b\}$ och $\{I, Å_b\}$.