Order of Residual Generators – Bounds and Algorithms

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Abstract

This contribution analyses residual generators that perfectly decouples disturbances in linear systems. The analysis focuses on the orders of the residual generators. Easily computed bounds on minimum and maximum order residual generators is derived and presented. An upper bound on the minimal row-degree is derived and is given directly by the number of measurements, number of linearly independent disturbances, and the number of states in the model. A lower bound is given by the minimum observability index of the model. An upper bound for the maximum order is the number of states in the model.

Keywords: fault diagnosis, residual generation, minimal indices, Kronecker indices, minimal order, disturbance decoupling

1 Introduction

This paper deals with supervision, or fault diagnosis, of computer controlled systems. The task of fault diagnosis is to, from known signals, i.e. measurements and control signals, detect and locate any faults acting on the system being supervised. A fundamental part of a *model based* diagnosis system is the *residual generator*. The residual generator filters known signals and generates a signal, the *residual*, that should be small (ideally 0) in the fault-free case and large when a fault is acting on the system. This signal can then be used as a fault indicator, signaling a faulty system.

To be able to produce a correct diagnosis in all operating conditions, influence from all disturbances on the residual need to be decoupled. Also, to facilitate fault isolation, not only disturbances need to be decoupled, but also a subset of the faults. By generating a set of such residuals where different subsets of faults are decoupled in each residual, fault isolation is possible. With this approach, the design of a residual generator becomes a decoupling problem. Further, only perfect decoupling of the disturbances is considered here, the issue of *approximate decoupling* associated with e.g. robust diagnosis is not considered.

This work is a study of the complexity of *linear* residual generators for *linear* systems with no model uncertainties where any faults and disturbances acting on the system are modeled as input *signals*. Of particular interest is the minimum complexity of residual generators. The reason for the interest in the minimal order property of the residual generator is primarily that we want to depend on the model as little as possible. A low order usually implies that only a small part of the model is utilized. Since all parts of the model have errors, this further means that few model errors will affect the residual. Also, lower complexity of the residual generator means easier implementation and less on-line computational burden. The following small example will highlight this issue. Consider a linear system with two sensors, one actuator, and a modeled sensor fault in the second sensor.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s+a} \\ \frac{1}{(s+b)(s+a)} \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f$$

The model consists of two model parameters, a and b. To detect the fault, a second order residual

$$r_1 = y_2 - \frac{1}{(s+a)(s+b)} u$$

can be used. Examining the expression gives that the residual relies on the accuracy of both model parameters a and b. Using straightforward manipulations of model equations, it is possible to derive a new, *first order* residual:

$$r_2 = \frac{1}{s+b} y_1 - y_2$$

As can be seen, residual r_2 only depends on the accuracy of parameter b. Thus, a lower order residual generator resulted in a residual generator less dependant on the model accuracy. Here, in this example, even complete invariance of model accuracy of parameter a was achieved. This is not a general result, model dependancy does not always decrease with the order. However, if the model has such a property, systematic utilization of low-order residual generators is desirable.

In Section 2, a few basic results from standard theory of polynomial matrices is presented. A brief description of the residual generation problem for linear systems is presented in Section 3, a more thorough description can be found in e.g. (Frisk 1998). The main results on the complexity of residual generators is given in Section 4. Discussions and examples on the use of these bounds are presented in Section 5.

2 Preliminaries

This paper relies on established theory on polynomial matrices, polynomial/rational vector spaces, and polynomial bases for these spaces (Kailath 1980, Forney 1975, Chen 1984). The main notions used, are presented in this section.

The *row-degree* of a row vector of polynomials is defined as the largest polynomial degree in the row-vector. In this paper, *polynomial bases* and *orders* of polynomial bases are of special interest. A polynomial basis is here represented by a polynomial matrix where the rows are the basis vectors. The *order* of a polynomial basis $F(s)$ is defined as the sum of the its row-degrees. A *minimal polynomial basis* for a rational vector-space $\mathcal{F}$ is then any polynomial basis that minimizes this order.

A matrix $F(s)$ is *irreducible* if and only if $F(s)$ has full rank for all s. A matrix $F(s)$ can always be written as

$$F(s) = S(s)D_{hr} + L(s)$$

where $S(s) = \text{diag}\{s^{\mu_i}, i = 1, \dots, p\}$, D_{hr} is the *highest-row-degree coefficient matrix*, μ_i is the row-degrees, and $L(s)$ is the rest term with row degrees strictly less than μ_i. A matrix is *row-reduced* if its highest-row-degree coefficient matrix D_{hr} has full row rank.

In addition to these definitions, the following theorems will be used:

Theorem 1 (Forney,1975). *A polynomial basis exists for any rational vector-space $\mathcal{F}$.*

Theorem 2 (Kailath,1980, Theorem 6.5-10). *The rows of a matrix $F(s)$ form a minimal polynomial basis for the rational vector space they generate, if and only if $F(s)$ is irreducible and row-reduced.*

Theorem 3 (Kailath,1980, p.401). *If the rows of $F(s)$ form an irreducible polynomial basis for a space $\mathcal{F}$, then all polynomial row vectors $x(s) \in \mathcal{F}$ can be written $x(s) = \phi(s)F(s)$ where $\phi(s)$ is a* polynomial *row vector.*

Theorem 4 (Kailath,1980). *For any linear matrix pencil $A - sB$, it is possible to find constant, square, and nonsingular matrices U and V such that*

$$U(A - sB)V =$$
$$= \textit{block diag}\{L_{\mu_1}, \dots, L_{\mu_\alpha}, \widetilde{L}_{\nu_1}, \dots, \widetilde{L}_{\nu_\beta}, sJ - I, sI - F\}$$

where

1. *F is in Jordan form*
2. *J is a nilpotent Jordan matrix*
3. *$\widetilde{L}_\nu$ is a $(\nu + 1) \times \nu$ matrix of the form*

$$\underbrace{\begin{bmatrix} -s & & & \\ 1 & -s & & \\ & 1 & & \\ & & \ddots & \\ & & & -s \\ & & & 1 \end{bmatrix}}_{\nu} \left.\vphantom{\begin{bmatrix} -s \\ 1 \\ 1 \\ \ddots \\ -s \\ 1 \end{bmatrix}}\right\} \nu + 1$$

4. *$L_\mu = \widetilde{L}_\mu^T$*

The $\{\nu_i\}$ and $\{\mu_i\}$ are called left and right Kronecker indices and are of particular interest in this paper.

Note: All matrices, besides $\widetilde{L}_\nu$ have full row-rank, $\widetilde{L}_\nu$ will therefore characterize the left null-space structure of the pencil. It is also easy to check that the left null-space of $\widetilde{L}_\nu$ is given by

$$v(s) = [1\ s\ \cdots\ s^\nu]$$

i.e, the degree of the (left) null-space vectors is directly given by the (left) Kronecker indices.

3 Linear Residual Generation

This section is a brief presentation of the linear residual generation problem. All derivations are performed in the continuous-time case but the corresponding results for the discrete-time case can be obtained by substituting s by z and improper by non-causal.

The systems studied in this work are assumed to be on the form

$$y = G_u(s)u + G_d(s)d + G_f(s)f \tag{1}$$

where y is the measurement vector, u is the vector of known inputs to the system, d is the disturbance vector, and f is the vector of faults. The symbols k_u, k_d, k_f, and m will be used to denote the number of control signals, disturbances, faults, and measurements respectively. A general linear residual generator can be written

$$r = Q(s)\begin{pmatrix} y \\ u \end{pmatrix} \tag{2}$$

i.e. $Q(s)$ is a multi-dimensional transfer matrix with known signals y and u as inputs and a *residual* as output. Here, $Q(s)$ is assumed to be a single output filter.

Definition 1. *The filter* $Q(s)$ *in* (2) *is a residual generator if and only if* $r = 0$ *for all* d *and* u *when* $f = 0$.

Note that to be able to detect faults, it is also required that $r \neq 0$ when $f \neq 0$.

Inserting (1) into (2) gives

$$r = Q(s)\begin{bmatrix} G_u(s) & G_d(s) \\ I & 0 \end{bmatrix}\begin{bmatrix} u \\ d \end{bmatrix} + Q(s)\begin{bmatrix} G_f(s) \\ 0 \end{bmatrix} f$$

To make $r(t) = 0$ when $f(t) = 0$, it is required that disturbances and the control signal are *decoupled*, i.e. for $Q(s)$ to be a residual generator, it must hold that

$$Q(s)\begin{bmatrix} G_u(s) & G_d(s) \\ I & 0 \end{bmatrix} = 0$$

This implies that $Q(s)$ must belong to the left null-space of

$$M(s) = \begin{bmatrix} G_u(s) & G_d(s) \\ I & 0 \end{bmatrix} \tag{3}$$

This null-space is denoted $\mathcal{N}_L(M(s))$. Therefore, the matrix $Q(s)$ need to fulfill two requirements: belong to the left null-space of $M(s)$ to ensure disturbance decoupling *and* have good fault sensitivity properties. If, in a first step of the design, *all* $Q(s)$ that fulfill the first requirement are found, then in a second step a single $Q(s)$ with good fault sensitivity properties can be selected. Thus, in a first step of the design, f or $G_f(s)$ need not be considered. The problem is then to find a basis for all rational $Q(s) \in \mathcal{N}_L(M(s))$. In (Nyberg & Frisk 1999), this problem is investigated in detail and it is showed how a minimal *polynomial* basis for this space can be computed. The main result is repeated in Section 3.1. In this paper, it is assumed that such a basis can be computed and is denoted $N_M(s)$.

The second and final design-step is to use the polynomial basis $N_M(s)$ to form the residual generator. The minimal polynomial basis $N_M(s)$ is by definition irreducible according to Theorem 2, and then according to Theorem 3, all decoupling *polynomial* vectors $F(s)$ can be parameterized as

$$F(s) = \phi(s)N_M(s) \tag{4}$$

where $\phi(s)$ is a polynomial vector of suitable dimension. The parameterization vector $\phi(s)$ can for example be used to shape the fault-to-residual response or simply to select one row in $N_M(s)$. Since $N_M(s)$ is a basis, the parameterization vector $\phi(s)$ have minimal number of elements, i.e. a minimal parameterization.

When a decoupling polynomial vector $F(s)$ has been selected for implementation to form a residual generator, it must be made realizable since a polynomial vector is improper and thus not realizable. A realizable rational transfer function $Q(s)$, i.e. the residual generator, can be found as

$$Q(s) = c^{-1}(s)F(s) \tag{5}$$

where the scalar polynomial $c(s)$ has greater or equal degree compared to the row-degree of $F(s)$. The degree constraint is the only constraint on $c(s)$. This means that the dynamics, i.e. poles, of the residual generator $Q(s)$ can be chosen freely, e.g. to impose a low-pass characteristic of the residual generator to filter out noise or high frequency model uncertainties. This also means that the minimal order of a realization of a decoupling filter is determined by the row-degrees of the minimal polynomial basis $N_M(s)$.

3.1 Computing the Basis

A main result in computing the basis $N_M(s)$ is to reduce the problem of computing a minimal polynomial basis for the left null-space of the *rational* matrix $M(s)$ into the problem of finding a minimal polynomial basis for the left null-space of a *polynomial* matrix. This is then a standard problem in the theory of polynomial matrices for which there exists standard tools readily available (*The Polynomial Toolbox 2.0 for Matlab 5* 1998).

The results of this section assumes that the system is written in state-space form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_d d \\ y &= Cx + D_u u + D_d d \end{aligned}$$

and n will be used to denote the number of states of a such a minimal realization.

Then an important matrix, which will be used extensively, the *system matrix* in state-space form with disturbances as inputs, can be formed. This matrix is denoted $M_s(s)$ and looks like:

$$M_s(s) = \begin{bmatrix} C & D_d \\ -(sI - A) & B_d \end{bmatrix} \tag{6}$$

Also, the following matrix, denoted P will be used:

$$P = \begin{bmatrix} I_m & -D_u \\ 0_{m\times n} & -B_u \end{bmatrix} \tag{7}$$

Then, the main theorem used here can be stated as:

Theorem 5 (Frisk,1998;Nyberg,1999). *Let* $V(s)$ *be a minimal polynomial basis for* $N_L(M_s(s))$ *and let the pair* $\{A, [B_u\ B_d]\}$ *be controllable. Then it holds that* $W(s) = V(s)P$ *is a minimal polynomial basis for* $M(s)$.

Remark 1: This theorem shows how the system matrix $M_s(s)$ is central in computing and analyzing the basis $N_M(s)$ and motivates its use in subsequent sections.
Remark 2: Since $N_M(s) = N_{M_s}(s)P$ where P is a constant matrix, it is clear that the the row-degrees of basis $N_M(s)$ is less or equal to the row-degrees of $N_{M_s}(s)$. In the next section, this observation will be strengthened and it is proved that the row-degrees of $N_M(s)$ and $N_{M_s}(s)$ are equal.
Remark 3: If the system model is given on transfer function form, a similar procedure found based on a right Matrix Fraction Description (MFD) of the system model can be used to transform the problem into a purely polynomial problem (Nyberg & Frisk 1999).

4 Row-Degrees of Basis

As discussed in the previous section, the row-degrees of a minimal polynomial basis for $\mathcal{N}_L(M(s))$ is closely connected with the order of the analytical relation used in the residual generator, and also the order needed to implement the residual generator. In this section, easily computed bounds on the minimal and maximal row-degrees of the basis $N_M(s)$ are derived.

Before further analysis on row degrees is made, a lemma is needed that shows that examining row-degrees of a basis for the relatively unstructured matrix $M(s)$ can be performed by examining the row degrees of the, structurally, much simpler system matrix $M_s(s)$. The primary property of $M_s(s)$ that makes it suitable for analysis is the fact that it has degree 1, i.e. it is a matrix pencil.

Lemma 1. *The row-degrees of a minimal polynomial basis for* $\mathcal{N}_L(M(s))$ *is equal to the row-degrees of a minimal polynomial basis for* $\mathcal{N}_L(M_s(s))$*, where* $M_s(s)$ *is the system matrix with the pair* $\{A, [B_u\ B_d]\}$ *controllable.*

Proof. Let $V(s)$ be a minimal polynomial basis for $\mathcal{N}_L(M_s(s))$ and partition $V(s) = [V_1(s)\ V_2(s)]$ according to the partition of $M_s(s)$. Since $V(s) \in \mathcal{N}_L(M_s(s))$, it holds that

$$V_1(s)C = V_2(s)(sI - A) = sV_2(s) - V_2(s)A$$

Also, since each row degree of $sV_2(s)$ is strictly greater than the corresponding row-degree of $V_2(s)A$, it holds that for each row i

$$\begin{aligned} \text{row-deg}_i\ sV_2(s) = \text{row-deg}_i\ V_2(s) + 1 = \\ = \text{row-deg}_i\ V_1(s)C \end{aligned}$$

The above equation can be rearranged to

$$\text{row-deg}_i\ V_2(s) < \text{row-deg}_i\ V_1(s)C \leq \text{row-deg}_i\ V_1(s) \tag{8}$$

i.e. $\text{row-deg}_i\ V(s) = \text{row-deg}_i\ V_1(s)$. From the definition of P it follows that

$$\begin{aligned} [W_1(s)\ W_2(s)] = V(s)P = \\ = [V_1(s) \quad (-V_1(s)D_u - V_2(s)B_u)] \end{aligned} \tag{9}$$

Equations (8) and (9) directly give

$$\text{row-deg}_i\ W(s) = \text{row-deg}_i\ V_1(s) = \text{row-deg}_i\ V(s),$$

i.e. the row degrees of $W(s)$ and $V(s)$ are equal. According to Theorem 5, $W(s)$ and $V(s)$ are minimal polynomial bases for $\mathcal{N}_L(M(s))$ and $\mathcal{N}_L(M_s(s))$ respectively and the lemma follows immediately. ■

Much of the structure of a matrix pencil is revealed by the Kronecker Canonical Form, given by Theorem 4. Specifically, the degrees of $N_{M_s}(s)$ is directly given by the left Kronecker indices[1] which can be extracted directly from a pencil on KCF. However, transferring a general pencil to KCF is a numerically tricky operation. It is therefore desired to have easily computed bounds or numerically stable algorithms for calculating these indices. In Section 4.1, bounds for the minimum and maximum row-degree of $N_M(s)$ are given. Section 4.2 gives pointers to some algorithms, that can be used to calculate the row-degrees without actually computing the basis.

[1] The Kronecker indices is sometimes called minimal indices.

4.1 Bounds on row-degrees

This section primarily analyzes the minimal row-degree $\rho_{\min}$, of the basis, since $\rho_{\min}$ is closely connected to the minimum complexity of a residual generator. However, before bounds on the minimum row-degree is derived, an upper bound on all row-degrees of a basis is directly given by the following theorem:

Theorem 6 (Nyberg,1999). *A matrix whose rows form a minimal polynomial basis for $\mathcal{N}_L(M(s))$ has all row-degrees $\leq n$.*

Now, an upper and a lower bound on the minimum row-degree is derived. First, a lower bound is derived, given by the following theorem:

Theorem 7. *A lower bound for the minimal row-degree ρ_{min} of a basis for $\mathcal{N}_L(M(s))$ is given by the minimal observability index of the pair (A, C).*

For the proof of this theorem, and other theorems to follow, the following lemma is needed. Denote

$$\widetilde{M}_\rho = \underbrace{\left.\begin{bmatrix} Q & R & & & \\ & Q & R & & \\ & & \ddots & \ddots & \\ & & & Q & R \end{bmatrix}\right\} (\rho+1)(m+n)}_{(\rho+2)(n+n_d)}$$

where $M_s(s) = Q + sR$ and Q,R are constant matrices. Then,

Lemma 2. *The space $\mathcal{N}_L(M_s(s))$ contains a ρ-degree polynomial vector if and only if $\widetilde{M}_\rho$ does not have full row rank.*

Proof. Let $F(s)$ be a ρ-degree polynomial matrix in $\mathcal{N}_L(M_s(s))$. Then it holds that

$$0 = F(s)M_s(s) = (F_0 + F_1 s + \cdots + s^\rho F_\rho)M_s(s) =$$
$$= [F_0\ F_1 \cdots F_\rho] \begin{bmatrix} M_s(s) \\ sM_s(s) \\ \vdots \\ s^\rho M_s(s) \end{bmatrix} = \widetilde{F}\widetilde{M}_\rho \begin{bmatrix} I \\ sI \\ \vdots \\ s^\rho I \end{bmatrix}$$

From the equation above it is clear that a ρ-degree polynomial $F(s)$ is in $\mathcal{N}_L(M_s(s))$ if and only if $\widetilde{F}\widetilde{M}_\rho = 0$. The lemma follows directly because such a $\widetilde{F}$ can only exist if $\widetilde{M}_\rho$ does not have full row-rank ∎

Remark: A similar result, given in a geometrical setting, can be found in (Karcanias & Kalogeropoulos 1988).

Return to the proof of Theorem 7.

Proof. Denote the system matrix without disturbances with $M_s^{(nd)}(s)$, i.e.

$$M_s^{(nd)}(s) = \begin{bmatrix} C \\ sI - A \end{bmatrix}$$

It is well known (Kailath 1980, p. 413), that the row-degrees of a minimal polynomial basis for the left null-space of $M_s^{(nd)}(s)$ equal the observability indices of the pair (A, C). Let $c_{\min}$ be the minimum observability index of (A, C). Then, according to Lemma 2, $c_{\min}$ is the lowest ρ such that $\widetilde{M}_\rho^{(nd)}$ does not have full row-rank. Let

$$Q = [Q_1\ Q_2] = \begin{bmatrix} C & D_d \\ A & B_d \end{bmatrix} \quad \text{and} \quad R = [R_1\ R_2] = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}$$

Then, by a trivial column reordering, $\widetilde{M}_\rho$ can be written on the form

$$\widetilde{M}_\rho = \begin{bmatrix} Q_1 & R_1 & & & & Q_2 & R_2 & & & \\ & Q_1 & R_1 & & & & Q_2 & R_2 & & \\ & & \ddots & & & & & \ddots & & \\ & & & Q_1 & R_1 & & & & Q_2 & R_2 \end{bmatrix} P$$
$$= [\widetilde{M}_\rho^{(nd)}\ \star]P$$

where P is a square, full rank pivoting matrix. From the equation above, it is clear that if $\widetilde{M}_\rho^{(nd)}$ has full row-rank, then also $\widetilde{M}_\rho$ has full row-rank. Also, for all $\rho < c_{\min}$, $\widetilde{M}_\rho^{(nd)}$ and thereby also $\widetilde{M}_\rho$, will have full row-rank. The theorem then follows directly from Lemma 2, i.e. there exists no ρ-degree polynomial in $\mathcal{N}_L(M_s(s))$ where $\rho < c_{\min}$. ∎

Remark: This result can also be found, without proof, in (Ding, Ding & Jeinsch 1998).

Theorem 8. *An upper bound for the minimal row-degree ρ_{min} of a basis for $\mathcal{N}_L(M(s))$ is given by*

$$\rho_{min} \leq \lfloor \frac{n + n_d}{m - n_d} \rfloor$$

where

$$n_d = \text{Rank} \begin{pmatrix} B_d \\ D_d \end{pmatrix}$$

is the number of linearly independent disturbances. The $\lfloor \cdot \rfloor$ operator is the floor operation.

Proof. Let A, B_u, B_d, D_u, and D_d be a minimal state-space realization of $[G_u(s)\ G_d(s)]$. If $n_d < k_d$, i.e. there exist linear dependencies between disturbances, rewrite the system description with a new set of n_d linearly independent disturbances. That is, find $\widetilde{B}_d$ and $\widetilde{D}_d$ with dimensions $n \times n_d$ and $m \times n_d$ respectively so that

$$\mathrm{Im}\begin{pmatrix} B_d \\ D_d \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \widetilde{B}_d \\ \widetilde{D}_d \end{pmatrix}$$

and use these in the state-space description. Now, using Lemma 1 and 2 it is clear that a ρ-degree polynomial vector is in $\mathcal{N}_L(M(s))$ if and only if $\widetilde{M}_\rho$ does not have full row rank. A sufficient condition for $\widetilde{M}_\rho$ not to have full row-rank is that the number of rows is larger than the number of columns, i.e.

$$(\rho+1)(m+n) > (\rho+2)(n+n_d)$$

Straightforward manipulations of the inequality results in

$$\rho > \frac{n+n_d}{m-n_d} - 1$$

Note that $m - n_d > 0$ is a necessary condition for the existance of a residual generator. Therefore, the smallest integer ρ that fulfills the inequality is $\lfloor \frac{n+n_d}{m-n_d} \rfloor$ which completes the proof. ∎

The result of Theorem 8 is useful when selecting the set of faults that are to be decoupled in the residual, i.e. when shaping the fault isolation properties. This theorem gives direct access to information on the expected complexity of the resulting residual generator, thereby making it possible to estimate the complexity of all residual generators without actually performing the designs. When shaping the isolation structure, i.e. selecting which and how many faults that are to be decoupled in each residual, the designer controls the quantity $m - n_d$. It holds that

$$\frac{n+n_d}{m-n_d} = \frac{n+m}{m-n_d} - 1$$

i.e. the upper bound decreases as $\propto 1/x$ with the designer controlled quantity $m - n_d$. Since the decrease is quite rapid, the complexity gain can be substantial, especially for high order processes that are well equipped with sensors.

4.2 Calculation of row-degrees

It is well known (Kailath 1980, p. 413), that the row-degrees of a minimal polynomial basis for the left null-space of $M_s^{(nd)}(s)$ equal the observability indices of the pair (A, C). However, no such straightforward algorithm exists for the general case including disturbances.

Of course, one could calculate the basis as described in Section 3. However, there are reasons for computing the indices without actually computing the basis itself. By only computing the Kronecker indices, which is a smaller problem than actually computing the basis, it is reasonable to assume that this would pose a numerically easier problem[2]. Therefore, in numerically difficult problems, computing the indices in a first step and then using the indices in the algorithm that extracts the basis. This could be accomplished by restricting the search for basis vectors to vectors of the degrees given by the initial index calculation.

There exist a lot of literature and algorithms for computing the Kronecker indices of a general pencil, e.g. (Misra, Dooren & Varga 1994, Wang, Dorato & Davison 1975, Aling & Scumacher 1984, Kågström 1986).

5 Examples

This section contains one small example where the results are applied, followed by a discussion for a larger industrial application, a model of a military jet-engine.

Small Example

Consider a system given by the following transfer functions.

$$y = G_u(s)u + G_d(s)d = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \\ 0 \end{bmatrix} u + \begin{bmatrix} \frac{2}{s+4} \\ \frac{2}{s+4} \\ \frac{3s+12}{(s+4)(s+3)} \end{bmatrix} d$$

Which can be realized by a 4:th order state-space description, i.e. $n = 4$. Matrix $M(s)$ then becomes

$$M(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+4} \\ \frac{1}{s+2} & \frac{2}{s+4} \\ 0 & \frac{3s+12}{(s+4)(s+3)} \\ 1 & 0 \end{pmatrix}$$

Direct inspection of matrix $M(s)$ gives that it has rank 2 and 4 rows, i.e. the dimension of the left null-space is $4 - 2 = 2$. Using Theorems 6, 7, and 8 give that for the

[2]To the authors knowledge, no such investigation has been made.

maximum and minimum row-degrees ρ_{max} and ρ_{min} it holds that

$$1 \leq \rho_{min} \leq 2 \tag{10a}$$

$$\rho_{max} \leq 4 \tag{10b}$$

The lower bound on ρ_{min} is given by the observability indices of the pair (A, C) which can be calculated to $2, 1, 1$. Computing a basis gives

$$N_M(s) = \begin{bmatrix} -s^2 - 3s - 2 & s^2 + 3s + 2 & 0 & 1 \\ 3s + 3 & -4.5s - 9 & s + 3 & 1.5 \end{bmatrix},$$

i.e. $\rho_{min} = 1$ and $\rho_{max} = 2$, which confirms that the inequalities (10) holds.

Jet-Engine Model

A model of a jet-engine developed by Volvo Aero Corporation, Trollhättan, Sweden, is used in this example. A high-order non-linear model of the engine is used for analysis and control design. This model can also be used for diagnosis purposes. The model was linearized in a working point and the resulting model, after that non-controllable and non-observable modes are eliminated, is a 26:th order model. The model used includes 8 sensors and 4 actuators.

The model is numerically stiff since it models fast dynamics, such as thermodynamics in small control volumes, and slow dynamics such as heating phenomena of metal. The largest time-constant in the model is about 10^5 times larger than the smallest time constant. This, together with the high-order, makes the model numerically sensitive which demands good numerical properties of the design algorithm.

In the design example, faults in sensors and actuators are considered. A residual that indicates a sensor failure is to be designed, i.e. all 4 actuator faults are to be decoupled. Using Theorem 8, it is clear that there exists residual generators with degree less than $\lfloor \frac{26+4}{8-4} \rfloor = 7$, which is significantly less than system order. Worth noting is how this limit depends on n_d. If a residual were to be designed that decoupled only one fault, i.e. $n_d = 1$, then the upper bound on the minimum degree residual generator would be as low as 3. This shows how it is possible to trade isolation properties for simpler fault detection filters.

It is also worth noting that, a design method not considering the order of the resulting residual generator easily results in a residual generator of the same order as the process model, here 26. However, with the minimal polynomial basis approach presented in Section 3, a 4:th order residual generator was found which shows how the minimality property here results in a filter with substantially less order than the order of the design model. This order gain, i.e. reduced order of the residual generator, can be substantial, especially when using detailed, high-order design models.

As mentioned, this model poses a numerically difficult design problem. Calculating the row-degrees of the basis with two different methods resulted in two different sets of row-degrees according to the table below:

Method	row-degrees of basis
(Wang et al. 1975)	$\{3, 3, 4, 4\}$
Computing the basis	$\{4, 5, 5, 7\}$

When evaluating the obtained basis by multiplying $N_M(s)M(s)$, the product does not become exactly 0 due to finite precision arithmetic. It does however become close to zero, but the row-degrees does not match the degrees obtained with the method described in (Wang et al. 1975). Future work will show if the quality of the computed basis can be improved by using pre-calculated row-degrees as above.

6 Conclusions

This contribution analyses residual generators that perfectly decouple disturbances in linear systems. The residual generation problem is formulated and it is briefly shown that the residual generation problem is transformed into finding a basis for the left null-space of a *rational* matrix $M(s)$. The analysis focuses on the row-degrees of the basis that are shown to be closely related to the order of residual generators.

First it is shown how the degree analysis can be performed on the, structurally much simpler, system matrix $M_s(s)$ instead of the rational matrix $M(s)$. The main difference between the $M_s(s)$ and $M(s)$ is that $M_s(s)$ is a matrix pencil, i.e. a polynomial matrix of degree 1 and it is shown how the row-degrees of a basis for the left null-space of $M_s(s)$ are equal to the row-degrees of a basis for the left null-space of $M(s)$. Then, easily computed bounds on the row-degrees of such a minimal polynomial basis are derived. An upper bound on the minimal row-degree is derived and given directly by the number of measurements, number of linearly independent disturbances, and the number of states in the model. A lower bound is given by the minimum observability index of the model.

These bounds can help the designer to estimate complexity of the diagnosis system and also help to guide the numerical algorithms used to find solutions to the residual generation problem.

Design examples are included to illustrate the use of the bounds. A design is performed on a 26:th order model of a jet-engine. A design algorithm that does not explicitly address minimality issues will likely end up with a residual generator of the same order as the system model. With the derived bounds, it was clear that a 7:th order residual generator existed. Performing the design with the proposed algorithm, a 4:th order residual generator was found.

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