Observers for non-linear differential-algebraic systems

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Abstract

In this paper we consider design of observers for non-linear models containing both dynamic and algebraic equations, so called differential-algebraic equations (DAE) or descriptor models. The obsverver is formulated as a DAE and the main results of the paper include conditions that ensure local stability of the observer and also that the observer has index 1. Design methodology is presented and illustrated using a small simulation study.

Keywords: Non-linear, observer, differential-algebraic equations, descriptor system

1 Introduction and observer formulation

State observation for nonlinear ODE models is a standard problem and has been studied for quite some time and is still an active area of research. The focus of this work is to design observers for models containing both dynamic and algebraic equations, so called differential-algebraic equations (DAE) or descriptor models. State observation for linear DAEs has been studied by e.g. [9] using the Kalman filter. Non-linear DAEs are considered in e.g. [1] where an extension of the Extended Kalman Filter is used and also by [11], where the original DAE model is rewritten as an ODE on a restricted manifold [10]. Other works include [2] that uses linearization tecniques and [6] that, in addition to a linearization procedure, employs index reduction thechniques to cope with high index models. In [5] a Lyapunov based approach is used in the design of the observer.

Now, our approach is introduced and the observer structure is presented. Usually, an observer is formulated as an ODE, $\dot{\hat{x}} = k(\hat{x}, u, y)$, for some vector field k. However, this work is an extension of a work by Nikhoukhah [7; 8] where it is noted that the requirement that the observer must be formulated as an ODE can be relaxed to a class of index 1 DAEs. This is due to the fact that low index DAEs are no more difficult to integrate than ODEs [3]. First, we briefly present the idea proposed in [8] for ODE models, and then our observer formulation for DAE models.

Consider the state-space model given by

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^m$ the measurement vector and $u \in \mathbb{R}^k$ the known control input. An often used observer is then

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x},u) + g(\lambda)\\ 0 &= y - h(\hat{x}) + \lambda\end{aligned}$$

using a, perhaps not so common, formulation using a slack variable λ, i.e. the function $g(\lambda)$ is the observer feedback used to ensure stability of the estimator. In [8], such a formulation is used to to define a class of observers in the form

$$\dot{\hat{x}} = f(\hat{x},u) + h_x(\hat{x})^T\dot{\lambda} + G(\hat{x},u)\lambda \tag{1a}$$
$$0 = y - h(\hat{x}) \tag{1b}$$

This observer is, under some mild technical assumptions, shown to be a DAE of index 1. The observer has some connections to reduced observers but does not inherit the possibly poor noise properties of reduced order observers. A discussion on this and other properties of the observer can be found in [7; 8].

Here, a similar approach is adopted for designing state estimators for the following class of semi-explicit models

$$\dot{x}_1 = f(x_1,x_2,z,t) \tag{2a}$$
$$0 = h(x_1,x_2,z,t) \tag{2b}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are state-variables and $z \in \mathbb{R}^{n_z}$ the vector of known signals, and $h \in \mathbb{R}^m$. The vector z includes both measurements and control signals and possibly other known quantities. Equation (2b) can include both measurement equations and algebraic constraints. Exact conditions on f and h are given in Section 3 and it is assumed that the model (2) has index 1.

The observer formulation used here for estimating x_i in (2), based on the known z, is

$$\dot{\hat{x}}_1 = f(\hat{x}_1,\hat{x}_2,z,t) + F(t)\dot{\lambda} + G(t)\lambda \tag{3a}$$
$$0 = h(\hat{x}_1,\hat{x}_2,z,t) \tag{3b}$$

where $\lambda \in \mathbb{R}^r$ and $r = m - n_2$. The observer gains F and G are the available design variables, which have to be chosen such that the observer has index 1 and provides a stable state estimate.

The outline of the paper is as follows. First, Section 2 shows how to ensure that the observer has index 1 such that the numerical integration of the observer is generally possible. Secondly, local estimator stability is explored in Section 3. The design method is summarized and exemplified on a simulation example based on components in an air suspension system of a heavy duty truck, in Section 4.

2 Observer Index

The objective of this section is to give conditions on F such that the observer (3) is a DAE with index 1. Before we can do that, some auxiliary subspaces of $\mathbb{R}^{n_1}$ have to be introduced. First, define the space

$$\mathcal{V} = \{x_1 : \ (x_1, x_2)^T \in N(h_x) \text{ for some } x_2\} \tag{4}$$

This means that $\mathcal{V}$ is the truncation of the null space $N(h_x)$ to $\mathbb{R}^{n_1}$. The first lemma shows that the dimension of the space is preserved under this truncation.

Lemma 1. *If h_x has full row rank and h_{x_2} has full column rank, then* $\dim \mathcal{V} = \dim N(h_x)$.

Proof. Since h_x has full row rank and $r = m - n_2$, we have

$$N(h_x) = \text{span}\,\{x^1, \dots, x^{n_1-r}\} \tag{5}$$

where $\{x^1, \dots, x^{n_1-r}\}$ is a linearly independent set. Using the notation

$$x^i = \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}$$

it follows from the definition of $\mathcal{V}$ that

$$\mathcal{V} = \text{span}\,\{x_1^1, \dots, x_1^{n_1-r}\}$$

We have to prove that $x_1^1, \dots, x_1^{n_1-r}$ are linearly independent, so assume therefore that

$$\sum_i \mu^i x_1^i = 0 \tag{6}$$

It follows from (5) that

$$\sum_i \mu^i x^i \in N(h_x)$$

and consequently

$$h_{x_1}\left(\sum_i \mu^i x_1^i\right) + h_{x_2}\left(\sum_i \mu^i x_2^i\right) = 0$$

Using that h_{x_2} has full column rank and assumption (6) we obtain

$$\sum_i \mu^i x_2^i = 0$$

Together with assumption (6), this implies that

$$\sum_i \mu^i x^i = 0$$

It follows that $\mu^1 = \ldots = \mu^{n_1-r} = 0$, since $x^1, \dots, x^{n_1-r}$ are linearly independent. Hence $x_1^1, \dots, x_1^{n_1-r}$ are linearly independent as well, which proves the lemma. ∎

Let $\mathcal{W}$ be an algebraic complement of $\mathcal{V}$ in $\mathbb{R}^{n_1}$, i.e. $\mathcal{W}$ is a subspace such that each $u \in \mathbb{R}^{n_1}$ has a unique representation $u = v + w$ where $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Let P_v and P_w denote the associated projections defined by $v = P_v u$ and $w = P_w u$. Now we can state and prove the main result of this section.

Theorem 1. *Suppose that h_x has full row rank and that h_{x_2} has full column rank. If $F(t)$ is chosen so that*

$$\operatorname{Im} F(t) = \mathcal{W} \tag{7}$$

then observer (3) *has index* 1.

Proof. Differentiate (3b) with respect to t

$$h_{x_1}\dot{\hat{x}}_1 + h_{x_2}\dot{\hat{x}}_2 + h_z\dot{z} + h_t = 0$$

which combined with (3a) can be written as

$$\begin{bmatrix} I & 0 & -F \\ h_{x_1} & h_{x_2} & 0 \end{bmatrix} \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f + G\lambda \\ -h_z\dot{z} - h_t \end{pmatrix} \tag{8}$$

That the observer has index 1 is equivalent to that the matrix on the left hand side is invertible. It is therefore sufficient to show that the homogeneous problem

$$x_1 - F\lambda = 0 \tag{9a}$$

$$h_{x_1}x_1 + h_{x_2}x_2 = 0 \tag{9b}$$

only has the trivial solution $x_1 = 0$, $x_2 = 0$ and $\lambda = 0$. It follows from (9b) that $x_1 \in \mathcal{V}$, and $F\lambda \in \mathcal{W}$ according to (7). This together with (9a) implies that $x_1 = F\lambda = 0$ since $\mathcal{W}$ is an algebraic complement of $\mathcal{V}$. Moreover $\lambda = 0$ since $F\lambda = 0$ and F has full column rank. Finally, $x_1 = 0$ and (9b) implies that $h_{x_2}x_2 = 0$ and consequently $x_2 = 0$ since h_{x_2} has full column rank. This proves that the matrix is invertible and that the index of the DAE is equal to 1. ∎

3 Stability Analysis

Given that F is chosen according to Theorem 1, we here present results that give conditions on G such that local observer stability is ensured. The main result of this section is Theorem 2 where it is shown that, under certain conditions, local stability of the non-linear DAE can be deduced from the stability of the linearizations of the error dynamics in either Lemma 2 or Lemma 3. Lemma 2 presents a straightforward expansion of the estimation error about the origin. In Lemma 3 a change of variables is introduced in this expansion and the stability problem is reduced to study an ordinary differential equation.

Lemma 2. *Assume that f and h have bounded first and second order partial derivatives with respect to x. If the estimation error $\tilde{x}_i = x_i - \hat{x}_i$ is sufficiently small, then*

$$\begin{bmatrix} I & 0 & F \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{bmatrix} f_{x_1} & f_{x_2} & -G \\ h_{x_1} & h_{x_2} & 0 \end{bmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \lambda \end{pmatrix} + \mathcal{O}(\|\tilde{x}\|^2) \tag{10}$$

Proof. The result is obtained by expanding the functions f and h about $(\hat{x}_1, \hat{x}_2)$.

$$\begin{aligned}
\dot{\tilde{x}}_1 &= f(x_1, x_2, z, t) - f(\hat{x}_1, \hat{x}_2, z, t) - F(t)\dot{\lambda} - G(t)\lambda \\
&= f_{x_1}(\hat{x}_1, \hat{x}_2, z, t)\tilde{x}_1 + f_{x_2}(\hat{x}_1, \hat{x}_2, z, t)\tilde{x}_2 - F(t)\dot{\lambda} - G(t)\lambda + \mathcal{O}(\|\tilde{x}\|^2) \\
0 &= h(x_1, x_2, z, t) - h(\hat{x}_1, \hat{x}_2, z, t) \\
&= h_{x_1}(\hat{x}_1, \hat{x}_2, z, t)\tilde{x}_1 + h_{x_2}(\hat{x}_1, \hat{x}_2, z, t)\tilde{x}_2
\end{aligned}$$

■

In the stability analysis we make the following assumptions:

Assumptions 1. Let x denote the solution of (2).

- $(h_x h_x^T)^{-1}$, $(h_{x_2}^T h_{x_2})^{-1}$ exist and are bounded in a neighbourhood of (x, z) uniformly in t.
- The functions f and h have bounded first and second order partial derivatives with respect to x in a neighbourhood of (x, z) uniformly in t.
- F, $\dot{F}$, G, $(F^T F)^{-1}$, P_v and P_w are bounded uniformly in t.

Note that the first assumption imply that h_x has full row rank and h_{x_2} has full column rank which in turn implies that the model (2) has index 1. The geometrical interpretation of the conditions on the projections P_v and P_w is that the angle between the subspaces $\mathcal{V}$ and $\mathcal{W}$ is bounded from below.

With the objective to reduce the stability problem into a study of an ordinary differential equation, we introduce a change of variables. The following transformation is considered:

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \lambda \end{pmatrix} = Q \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \tag{11}$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} P_v & 0 & -F \\ P_e & I & 0 \\ (F^T F)^{-1} F^T P_w & 0 & I \end{bmatrix} \tag{12}$$

Here P_v and P_w are the projections introduced in the previous section and P_e is a matrix such that

$$\operatorname{Im} \begin{bmatrix} P_v \\ P_e \end{bmatrix} = N(h_x)$$

The choice of new state-variables is motivated by the following result.

Lemma 3. *Assume that Assumptions 1 are fulfilled and that ξ is defined by the transformation* (11) *with the transformation matrix* (12)*. Then*

$$\dot{\xi}_1 = A_1(t)\xi_1 + \mathcal{O}(\|\xi_1\|^2)$$

with

$$A_1 = f_{x_1} Q_{11} + f_{x_2} Q_{21} - G Q_{31} - \dot{Q}_{11} - F \dot{Q}_{31}$$

and

$$\xi_i = \mathcal{O}(\|\xi_1\|^2), \quad i = 2, 3$$

Proof. Consider the first equation in (10):

$$\dot{\tilde{x}}_1 + F\dot{\lambda} = f_{x_1}\tilde{x}_1 + f_{x_2}\tilde{x}_2 - G\lambda + \mathcal{O}(\|\tilde{x}\|^2) \tag{13}$$

Using the transformation (11), the left-hand side of (13) can be rewritten as

$$\begin{aligned}\dot{\tilde{x}}_1 + F\dot{\lambda} &= (Q_{11} + FQ_{31})\dot{\xi}_1 + (Q_{12} + FQ_{32})\dot{\xi}_2 + (Q_{13} + FQ_{33})\dot{\xi}_3 \\ &\quad + (\dot{Q}_{11} + F\dot{Q}_{31})\xi_1 + (\dot{Q}_{12} + F\dot{Q}_{32})\xi_2 + (\dot{Q}_{13} + F\dot{Q}_{33})\xi_3\end{aligned}$$

The transformation Q has been chosen so that $Q_{11}+FQ_{31} = I$, $Q_{12}+FQ_{32} = 0$ and $Q_{13} + FQ_{33} = 0$.

Rewriting the right-hand side of (13) we get

$$\begin{aligned}f_{x_1}\tilde{x}_1 + f_{x_2}\tilde{x}_2 - G\lambda &= (f_{x_1}Q_{11} + f_{x_2}Q_{21} - GQ_{31})\xi_1 \\ &\quad + (f_{x_1}Q_{12} + f_{x_2}Q_{22} - GQ_{32})\xi_2 \\ &\quad + (f_{x_1}Q_{13} + f_{x_2}Q_{23} - GQ_{33})\xi_3\end{aligned}$$

The transformation Q has the bounded inverse

$$Q^{-1} = \begin{bmatrix} I & 0 & F \\ -P_e & I & -P_eF \\ -(F^TF)^{-1}F^TP_w & 0 & 0 \end{bmatrix}$$

and the remainder $\mathcal{O}(\|\tilde{x}\|^2)$ can therefore be replaced by $\mathcal{O}(\|\xi\|^2)$.

Summing up, we have shown that (13) can be rewritten as

$$\dot{\xi}_1 = A_1\xi_1 + A_2\xi_2 + A_3\xi_3 + \mathcal{O}(\|\xi\|^2) \tag{14}$$

where

$$A_i = f_{x_1}Q_{1i} + f_{x_2}Q_{2i} - GQ_{3i} - \dot{Q}_{1i} - F\dot{Q}_{3i}$$

It follows from Assumptions 1 that A_2 and A_3 are bounded.

Now, consider the second equation in (10):

$$0 = h_{x_1}\tilde{x}_1 + h_{x_2}\tilde{x}_2 + \mathcal{O}(\|x\|^2) \tag{15}$$

Using the transformation (11) we get

$$\begin{aligned}h_{x_1}\tilde{x}_1 + h_{x_2}\tilde{x}_2 &= (h_{x_1}Q_{11} + h_{x_2}Q_{21})\xi_1 \\ &\quad + (h_{x_1}Q_{12} + h_{x_2}Q_{22})\xi_2 \\ &\quad + (h_{x_1}Q_{13} + h_{x_2}Q_{23})\xi_3\end{aligned}$$

The transformation Q is chosen so that $h_{x_1}Q_{11} + h_{x_2}Q_{21} = 0$ and (15) can be rewritten as

$$h_x \begin{pmatrix} -F\xi_3 \\ \xi_2 \end{pmatrix} = r$$

where $r = \mathcal{O}(\|\xi\|^2)$. We will complete the proof by showing that this implies that

$$\xi_i = \mathcal{O}(\|\xi_1\|^2), \quad i = 2, 3$$

As a first step it is shown that the subspace

$$\mathcal{W} \times \mathbb{R}^{n_2} = \{(x_1, x_2) : x_1 \in \mathcal{W}, x_2 \in \mathbb{R}^{n_2}\}$$

is an algebraic complement of $N(h_x)$ and that the associated projections onto $N(h_x)$ and $\mathcal{W} \times \mathbb{R}^{n_2}$ are bounded. The projections are

$$P_1 = \begin{bmatrix} P_v & 0 \\ P_e & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} P_w & 0 \\ -P_e & I \end{bmatrix}$$

which are both bounded. It is not difficult to verify that $P_1 + P_2 = I$, $\operatorname{Im} P_1 = N(h_x)$ and $\operatorname{Im} P_2 = \mathcal{W} \times \mathbb{R}^{n_2}$. It remains to show that the intersection of the subspaces only contains the zero vector. Assume therefore that

$$\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the vectors in on the left-hand side belong to $N(h_x)$ and $\mathcal{W} \times \mathbb{R}^{n_2}$ respectively. The first equation, $x_{11} + x_{21} = 0$, and $x_{11} \in \mathcal{V}$ and $x_{21} \in \mathcal{W}$ implies that $x_{11} = x_{21} = 0$, since $\mathcal{W}$ is an algebraic complement of $\mathcal{V}$. Moreover $h_{x_1} x_{11} + h_{x_2} x_{12} = 0$ and $x_{11} = 0$ which implies that $h_{x_2} x_{12} = 0$ and consequently $x_{12} = 0$. Combined with $x_{12} + x_{22} = 0$ this implies $x_{22} = 0$. This proves that $\mathcal{W} \times \mathbb{R}^{n_2}$ is an algebraic complement of $N(h_x)$.

Now we proceed with the estimates of ξ_2 and ξ_3. Introduce the auxiliary vector

$$\xi^{\perp} = h_x^T (h_x h_x^T)^{-1} r = \mathcal{O}(\|\xi\|^2)$$

By using $h_x \xi^{\perp} = r$, $P_1 + P_2 = I$ and $h_x P_1 \xi^{\perp} = 0$ we get

$$h_x P_2 \xi^{\perp} = r \tag{16}$$

It follows from the definition of P_2 that

$$P_2 \xi^{\perp} \in \mathcal{W} \times \mathbb{R}^{n_2}$$

Furthermore

$$h_x \begin{pmatrix} -F\xi_3 \\ \xi_2 \end{pmatrix} = r \tag{17}$$

and

$$\begin{pmatrix} -F\xi_3 \\ \xi_2 \end{pmatrix} \in \mathcal{W} \times \mathbb{R}^{n_2}$$

The equations (16) and (17) present two solutions of the linear equation $h_x x = r$. Both solutions are members of the same algebraic complement of the null space of h_x and therefore they have to coincide, and consequently

$$\begin{pmatrix} -F\xi_3 \\ \xi_2 \end{pmatrix} = P_2 \xi^{\perp} = \mathcal{O}(\|\xi\|^2)$$

This gives that

$$\xi_i = \mathcal{O}(\|\xi\|^2), \quad i = 2, 3$$

since $(F^T F)^{-1}$ is bounded. This implies that

$$\xi_i = \mathcal{O}(\|\xi_1\|^2), \quad i = 2, 3$$

which together with (14) proves the lemma. ■

It is not always the case that stability of a linearization of a non-linear DAE implies local stability of the original DAE. However, under Assumptions 1 either one of the linearizations

$$\begin{bmatrix} I & 0 & F \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{bmatrix} f_{x_1} & f_{x_2} & -G \\ h_{x_1} & h_{x_2} & 0 \end{bmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \lambda \end{pmatrix} \tag{18}$$

$$\dot{\xi}_1 = (f_{x_1} Q_{11} + f_{x_2} Q_{21} - G Q_{31} - \dot{Q}_{11} - F \dot{Q}_{31}) \xi_1 \tag{19}$$

from Lemma 2 and Lemma 3 can be used to ensure local observer stability.

Theorem 2. *Suppose Assumptions 1 are fulfilled, $\tilde{x}(0), \lambda(0)$ are sufficiently small, and that the linearized error dynamics* (18) *or* (19) *is asymptotically stable. Then $\tilde{x}$ and λ tend to* 0 *as t tends to infinity.*

Proof. According to Lemma 3 it holds that stability of

$$\dot{\xi}_1 = A_1(t)\xi_1 + \mathcal{O}(\|\xi_1\|^2)$$

implies local stability of the error dynamics. For ODEs, contrary to DAEs, it is always the case that stability of the linearization implies local stability of the non-linear ODE. Thus, local stability of error dynamics is implied by the stability of

$$\dot{\xi}_1 = A_1(t)\xi_1$$

i.e. stability of (19).

The change of variables from Lemma 3 in the linearization (18) gives

$$\dot{\xi}_1 = A_1(t)\xi_1 + A_2(t)\xi_2 + A_3(t)\xi_3 \tag{20a}$$

$$0 = h_x \begin{pmatrix} -F\xi_3 \\ \xi_2 \end{pmatrix} \tag{20b}$$

Following the same steps as in the proof of Lemma 3 it can be shown that (20b) implies $\xi_2 = 0$ and $\xi_3 = 0$. Thus, stability of (18) implies stability of (19) which completes the proof. ■

4 Design Summary and a Simulation Example

The section will briefly summarize the design procedure and apply the method on a small example inspired by an air suspension system in a truck and also provide some simulation results.

4.1 Design summary

Initially it is assumed that the model is in the form (2) and that Assumption 1 is fulfilled, then the observer is given by (3). There are two design variables in the observer, the observer gains F and G, where the former is used to ensure that the observer has index 1 and the latter is used to ensure stability of the state estimator. The design of F and G can be done in two steps:

- The design of F is done using Theorem 1 by computing an algebraic complement $\mathcal{W}$ to the space $\mathcal{V}$ defined in (4) and letting $\operatorname{Im} F(t) = \mathcal{W}$.

- When F has been determined, G is chosen such that the conditions in Theorem 2 are satisfied. There are essentially two ways to perform the design.

  A first design approach is to directly use a linearization of the observer dynamics which results in a linear DAE (18) for which linear DAE observer methodology can be applied.

  A second approach is to compute a transformation, given in the proof of Theorem 2, which finds an ODE (19) for which any observer design technique can be employed. It is also clear from the proof of Theorem 2 that a constant F is advantageous since then both $\dot{Q}_{11}$ and $\dot{Q}_{31}$ in (19) vanish which makes the design easier.

4.2 Small simulation example

A principle sketch of the example system is shown in Figure 1. The system consists of a bellows and interconnected components. Basic operation is such

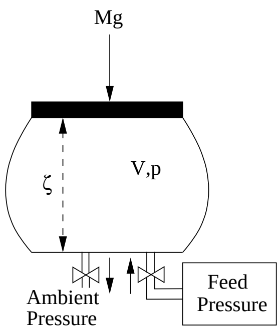


Figure 1: Principle sketch of the bellows.

that using a height sensor, a control system, actuating valves and a pump, the bellows is controlled at a user controlled preset height. The model equations can be written as

$$M\ddot{\zeta} = -Mg + F_b(p,\zeta) - \mu\dot{\zeta} \tag{21a}$$

$$pV(p,\zeta) = m_{air}RT \tag{21b}$$

$$\dot{m}_{air} = u_1\frac{P_{feed}}{\sqrt{RT}}\Psi(\frac{p}{P_{feed}}) - u_2\frac{p}{\sqrt{RT}}\Psi(\frac{P_{atm}}{p}) \tag{21c}$$

$$0 = y - \zeta \tag{21d}$$

where ζ is the height of the bellows, M the mass load, μ a friction/damping coefficient, p the pressure inside the bellows, and m_{air} the mass of air inside the bellows. The signals u_1 and u_2 are control signals for valves letting air in and out of the bellows and y is the height measurement signal. The functions $F_b(p,\zeta)$ and $V(p,\zeta)$ are non-linear maps of the force and volume respectively of the bellows as a function of pressure and height. These nonlinear maps are provided by the bellows manufacturer and is obtained by mapping bellows characteristics in a test bench. The function $\Psi(\cdot)$ is a non-linear function that describes the flow in and out of the bellows past the valves, see [4, Appendix

C] for details. Here, the flow is modeled as compressible flow of a perfect gas through a venturi. In the simulations, the ratio of pressures before and after the valve is both above and below the critical pressure ratio. This means that both sonic and subsonic flow velocities are present and therefore a strong nonlinearity need to be considered.

It is straightforward to put the model in the form (2) using $z = (y, u_1, u_2)$ and the state variables

$$x_1 = (\zeta, \dot{\zeta}, m_{air})$$
$$x_2 = p$$

Note that there is no dynamic equation for the pressure p and that it is non-trivial to obtain an explicit expression of p from (21b) since p is included in the mapped function $V(p, \zeta)$.

For the design of observer gain F, Theorem 1 is used. First, observe that

$$h_x = \begin{bmatrix} x_2 \frac{\partial V}{\partial x_{11}} & 0 & -RT & \frac{\partial}{\partial x_2}(x_2 V) \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

clearly has full row-rank. For physical reasons it holds that $\frac{\partial}{\partial x_2}(x_2 V) > 0$, and it follows that h_{x_2}, i.e. the fourth column in h_x, has full column-rank. It is then straightforward to verify that the space $\mathcal{V}$ defined in (4) and an algebraic complement $\mathcal{W}$ are given by

$$\mathcal{V} = \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}, \; \mathcal{W} = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$$

Then, Theorem 1 gives that the observer gain F can be chosen as

$$F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which ensures that the observer has index 1. For the stability and design of observer gain $G = (g_1(u), g_2(u), g_3(u))$, the linearized dynamics is computed using Theorem 2 and (19). Thus, $A_1(t) = f_{x_1} P_v + f_{x_2} Pe - G(F^T F)^{-1} F^T P_w$ where

$$P_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$P_e = \begin{bmatrix} 0 & 0 & \frac{RT}{\frac{\partial}{\partial x_2}(x_2 V)} \end{bmatrix}$$

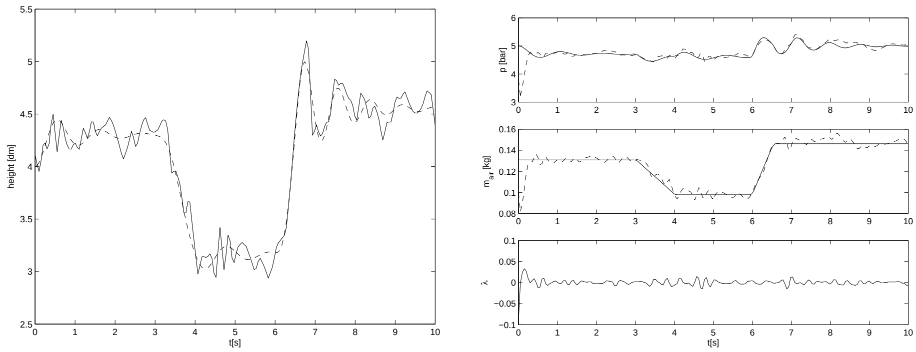
This gives that

$$A_1(t) = \begin{bmatrix} -g_1 & 1 & 0 \\ -g_2 & -\mu/M & q_1 \\ -g_3 & 0 & q_2 \end{bmatrix}$$

where q_1 and q_2 are defined as

$$
\begin{aligned}
q_1 =& \frac{RT}{M} \frac{\frac{\partial F_b}{\partial x_2}}{\frac{\partial}{\partial x_2}(x_2 V)} \\
q_2 =& \frac{\sqrt{RT}}{\frac{\partial}{\partial x_2}(x_2 V)} \Bigg(u_1 \Psi'(\frac{x_2}{P_{feed}}) \\
& + u_2 (P_{atm} \Psi'(\frac{P_{atm}}{x_2}) - \Psi(\frac{P_{atm}}{x_2})) \Bigg)
\end{aligned}
$$

Using this expression, the observer gain G can be determined by e.g. pole placement in a suitable operating point or using more elaborate schemes using, Kalman filters, gain scheduling techniques etc. In this small example, the observer gain G is determined by placing the observer dynamics poles in -10 in the operating point $\zeta_0 = 3.5$ dm and $p_0 = 5$ bar. To make the simulation a little more realistic, measurement noise is added and some modelling errors are introduced (+10% for the loaded mass M and -10% for the damping coefficient μ). Figure 2a shows the height of the bellows during simulation and also the measurement signal y to show the level of noise. The estimation of the height



(a) Height of bellows during simulation. The solid line is the measured height and the dashed the true height.

(b) Result of observer for the simulation. The solid lines are true values and dashed the observer estimate. The lowest plot shows the observer relaxation variable λ.

Figure 2: System and observer simulation.

ζ of the bellows is given by the measurement signal y in Figure 2-a since, according to the observer equation (3), the measurement equation is part of the algebraic constraints. For the simulation all DAEs were integrated using the Matlab solver `ode15s`. The example shows that, at least in the demonstrated case, how the designed observer provides good estimates in the presence of noise and significant modelling errors.

5 Conclusions

In this paper we have studied state estimation for semi-explicit differential-algebraic models. The proposed observer is formulated as a DAE. Conditions on the design parameters in the observer are derived in Theorem 1 such that the index of the observer is 1. This result ensures that we are able to integrate the observer easily. It is shown in Theorem 2 that we can use the linearization of the error dynamics to obtain local stability of the observer. This provides one possibility to design the observer by studying the linearized system and using available linear DAE techniques. An alternative way is to introduce a change of variables, which reduces the stability problem into a study of stability of an ODE. Therefore, general methods such as pole placement or gain scheduling techniques can be used.

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