Residual Generation in Linear Stochastic Systems - A Polynomial Approach

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Abstract

A polynomial design algorithm for innovation/residual generation for fault diagnosis is given. The class of systems considered is linear stochastic state-space and descriptor systems in both continuous and discrete time. A new class of residual generators based on stochastic models is defined and design algorithms are derived. The algorithms is based on standard operations such as null-space computation and spectral factorization of polynomial matrices for which numerically reliable implementations is readily available.

1 Introduction

This work deals with residual generation for fault diagnosis in linear systems. A residual is a fault-sensitive signal that is produced by filtering known signals, i.e. the control signals and measured signals. The residual should, ideally, be zero in the fault-free case regardless of any unknown disturbances and non-zero in case of a fault.

For deterministic systems, this is a well studied area (Chow and Willsky, 1984; Gertler, 1991; Nikoukhah, 1994) to name but a few. In previous works, (Frisk and Nyberg, 2001), design algorithms and analysis tools were developed based on polynomial methods, instead of parity-space based approaches or observer-based approaches. The polynomial approach proved to be, apart from a numerically sound design tool, very well suited to answer many fundamental questions regarding e.g. complexity of residual generators, simple parameterization of residual generators.

For stochastic linear systems, here noise affected linear systems, there is not as much work published. A common approach for these systems is to use Kalman-filters as residual generators which then produces residuals that is zero-mean and white with known covariance. The drawback of this approach is that systems subjected to unknown inputs which cannot, in any reasonable way, be modeled as random processes with known statistics, is not handled. This is often the case for the fault isolation task were a subset of the faults must be decoupled in the residual. Detailed stochastic information about fault signals is rarely available and often, just modeling the fault influence on the process is difficult enough. This means that the diagnosis decision should not be based on any residual that is "corrupted" by these unknown signals, i.e. they should be decoupled in the residual.

A fundamental contribution to this problem is given by Nikoukhah (1994) where *innovation filters* was defined. Here, the aim is to use and extend the polynomial methods that proved beneficial in the deterministic case to the stochastic case and address the problems posed in (Nikoukhah, 1994). Also, the problem formulation is extended to solve a more general design problem. As a consequence of the approach, extensions to handle also stochastic *descriptor systems* is immediate. Worth noting is that residual generation for diagnosis is not the same as a fault estimation problem, thus e.g. minimum variance estimation of fault signals is not necessarily a good idea.

The main algorithmic tool is *J-spectral co-factorization* which is shown to quite nicely handle the stochastic problem. Algorithms for spectral factorization of polynomial matrices has recently received much attention since it plays a fundamental role in the solution of polynomial $\mathcal{H}_\infty$ and $\mathcal{H}_2$-standard problems. Therefore, feasible and numerically appealing algorithms and implementations has been proposed (Kwakernaak and Šebek, 1994; Kwakernaak, 2000).

2 Problem formulation

The system under consideration is described by:

$$\dot{x} = Ax + B_u u + B_d d + B_n n + B_f f \tag{1a}$$
$$y = Cx + D_u u + D_d d + D_n n + D_f f \tag{1b}$$

where $y \in \mathbb{R}^m$ is the measurement vector, $u \in \mathbb{R}^{k_u}$ control signals, $d \in \mathbb{R}^{k_d}$ unknown disturbances, $f \in \mathbb{R}^{k_f}$ faults, $n \in \mathbb{R}^{k_n}$ noise, and A, B, C, and D are constant matrices of suitable dimensions. Further on, $G_u(s)$, $G_d(s)$, $G_f(s)$, and $G_n(s)$ will be used to denote transfer functions from u, d, f, n to y respectively. The difference between the disturbances d and the noise n is that the disturbances is assumed to have no stochastic description and must be decoupled while the noise is modeled as a white stationary stochastic process with unit covariance. The noise is not decoupled but is handled otherwise. From now on it is assumed that perfect decoupling of the noise n is not possible. A discussion on the singular case that arises when the noise is perfectly decouplable is included in (Frisk, 2001).

For linear models with no unknown inputs (i.e. no d), the innovations process associated with the Kalman filter is often used as a residual because of its zero-mean and whiteness properties in the fault-free case. Once the innovations is generated, the fault decision problem reduces to a whiteness test of the residual. Also, other more elaborate decision algorithms can be used based on more deep utilization of stochastic properties of the residual (Basseville and Nikiforov, 1993).

Nikoukhah (1994) included unknown inputs in a definition of residual generators for stochastic systems where the whiteness property of the residual is achieved without restricting the number of linearly independent residuals and thereby (possibly) limiting fault detectability properties.

Definition 1 (Innovation filter). *A finite-dimensional linear time-invariant system $Q(s)$ is called an innovation-filter for system* (1) *if it is stable with the least number of outputs such that, in the absence of failure,*

1. *its output*
$$r = Q(s)\begin{pmatrix} y \\ u \end{pmatrix}$$
is zero-mean, white and decoupled from u *and* d*,*

2. *if* $Q'(s)$ *is any finite-dimensional linear time-invariant system such that*
$$r' = Q'(s)\begin{pmatrix} y \\ u \end{pmatrix}$$
is decoupled from u *and* d*, then there exists a linear system* $L(s)$ *such that* $Q'(s) = L(s)Q(s)$*.*

Innovation filters does not always exist and this leads to a relaxation of the conditions of Definition 1 which will be shown to be useful.

Definition 2 (Whitening residual generator). *A stable and proper linear filter* $Q(s)$ *is a whitening residual generator for* (1) *if and only if when* $f \equiv 0$ *it holds that*

$$r = Q(s)\begin{pmatrix} y \\ u \end{pmatrix}$$

is zero mean and white for all u*,* d*.*

3 Introductory examples

Before going into detail, describing a design algorithm and existence conditions, two small illustrative examples are presented, illustrating the two cases when innovation filters/whitening residual generators do not exist.

3.1 Example 1: Zeros on the imaginary axis

Consider the model:

$$y = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} d + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Straightforward calculations gives that all disturbance decoupling residual generators $Q(s)$ can be parameterized by the rational parameter $\varphi(s)$ as

$$Q(s) = \varphi(s)[0 \;\; s(s+1) \;\; -1]$$

for which the fault-free internal form is given by

$$r = \varphi(s)s(s+1)n_2$$

Here it is clear that no strictly stable $\varphi(s)$ exists making r white, all because of the finite zero on the imaginary axis in the transfer function from n to r.

This also shows a link to non strongly detectable faults(Chen and Patton, 1994). A zero at $s = 0$ will appear in the transfer function from n to r if n enters the system in the same way as a non strongly detectable fault f which was the case in the example above.

3.2 Example 2: Infinite zeros

Consider the scalar system:

$$y = \frac{1}{s+1}u + f + \frac{1}{(s+2)^2}n$$

All residual generators $Q(s)$ can be written

$$Q(s) = \varphi(s)[s+1 \;\; -1]$$

for which the internal form is

$$r = \varphi(s)\frac{s+1}{(s+2)^2}n \tag{2}$$

It is clear that for r to be white $\varphi(s) = \frac{(s+2)^2}{s+1}$ which gives an improper, and thus non-realizable, residual generator

$$r = [(s+2)^2 \;\; -\frac{(s+2)^2}{s+1}]\begin{pmatrix} y \\ u \end{pmatrix}$$

And this was caused by the infinite zero of the transfer function $\frac{s+1}{(s+2)^2}$ in (2).

Now, with these cases in mind, a design algorithm is described in the next section.

4 Design algorithm

Previous works (Frisk and Nyberg, 2001) give that any residual generator $r = Q(s)\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ based on a deterministic model can be written as $Q(s) = \varphi(s)N_{M_s}(s)P_x$ where

$$M_s(s) = \begin{bmatrix} C & D_d \\ -(sI_{n_x} - A) & B_d \end{bmatrix} \quad \wedge \quad P_x = \begin{bmatrix} I_{k_m} & -D_u \\ 0_{n_x \times k_m} & -B_u \end{bmatrix} \tag{3}$$

and n_x is the number of states i.e. the size of x and $N_{M_s}(s)$ is a minimal polynomial basis for the left null-space of $M_s(s)$. All available design freedom lies in the choice of the rational row-vector/matrix $\varphi(s)$ which here will be used to realize whitening residual generators and innovation filters. One constraint on $\varphi(s)$ is that $Q(s)$ must be realizable and stable.

Now, existence conditions and design procedures will be derived. Due to space constraints, this presentation is focused on innovation filters and only briefly state results on whitening residual generators. For detailed results on whitening residual generators, see (Frisk, 2001).

Lemma 1. *A transfer matrix* $Q(s)$ *is an innovation filter for system* (1) *if and only if there exists a matrix* $\varphi(s)$ *such that*

$$Q(s) = \varphi(s)N_{M_s}(s)P_x$$

is proper, stable and it holds that

$$\forall s. H(s)H^T(-s) = \Psi$$

where $H(s) = \varphi(s)N_{M_s}(s)\begin{pmatrix} D_n \\ B_n \end{pmatrix}$*,* $\Psi \in \mathbb{R}^{r \times r}$ *is a constant full-rank matrix, and* $r = \dim \mathcal{N}_L(M_s(s))$*.* $\mathcal{N}_L(M_s(s))$ *denotes the left null-space of matrix* $M_s(s)$*.*

Proof. All disturbance decoupling residual generators can be written as

$$Q(s) = \varphi(s)N_{M_s}(s)P_x$$

Insertion of (1) into $r = Q(s)\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ gives

$$r = \varphi(s)N_{M_s}(s)P_x \begin{pmatrix} y \\ u \end{pmatrix} = \varphi(s)N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix} n \qquad (4)$$

Whiteness of r is equivalent to $\Phi_r(j\omega)$ constant for all ω which, since $\Phi_r(s)$ is rational, is equivalent to $\Phi_r(s)$ is constant for all s. The problem is assumed normalized such that $\Phi_n(s) = I$, then the spectrum $\Phi_r(s)$ can be written as

$$\Phi_r(s) = \varphi(s)N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}\begin{bmatrix} D_n \\ B_n \end{bmatrix}^T N_{M_s}^T(-s)\varphi^T(-s) = \\ = H(s)H^T(-s)$$

and the theorem follows immediately. ∎

Now, for sake of notational convenience, let $Z(s) \in \mathbb{R}^{m\times m}[s]$ denote

$$Z(s) = N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}\begin{bmatrix} D_n \\ B_n \end{bmatrix}^T N_{M_s}^T(-s) \qquad (5)$$

Then, the spectrum of r can be written

$$\Phi_r(s) = \varphi(s)Z(s)\varphi^T(-s)$$

This also implies that the assumption made in Section 1, that it is not possible to perfectly decouple the stochastic noise n, is equivalent to $Z(s)$ being full-rank. If $Z(s)$ would be rank deficient, there would exist a non-zero $\varphi(s)$ such that the spectrum of r would be identically 0, i.e. the noise would be perfectly decoupled. Therefore, in this section it is assumed, unless otherwise noted, that $Z(s)$ is full-rank. Further discussions on the case when $Z(s)$ is not full-rank is found in (Frisk, 2001).

Before the main result can be stated, a lemma characterizing the parameterization matrix $\varphi(s)$ in Lemma 1 is needed:

Lemma 2. *Assume $Z(s)$ full-rank. Then there exists a $\varphi(s)$ such that the linear time-invariant filter $Q(s) = \varphi(s)N_{M_s}(s)P_x$ produces white residuals if and only if $\varphi(s)$ can be written*

$$\varphi(s) = \eta(s)P^{-1}(s)$$

where $P(s)$ is a spectral co-factor of $Z(s)$ and $\eta(s)\eta^T(-s) = \Psi$ for some constant matrix Ψ.

Proof. The spectrum of r can be written

$$\Phi_r(s) = \varphi(s)Z(s)\varphi^T(-s) \qquad (6)$$

Note that $Z(s)$ is a p.h. polynomial matrix. Now, let $P(s)$ be a spectral co-factor and J a signature of $Z(s)$, i.e.

$$Z(s) = P(s)JP^T(-s) \qquad (7)$$

where $P(s)$ is a square, full-rank matrix with invariant zeros in the closed left half-plane. Since $Z(s)$ is assumed positive definite it has signature $J = I_m$. Insertion of (7) into (6) and denoting $\eta(s) = \varphi(s)P(s)$ gives

$$\Phi_r(s) = \varphi(s)P(s)JP^T(-s)\varphi^T(-s) = \eta(s)\eta^T(-s)$$

Thus, $\Phi_r(s)$ is constant for all s if and only if $\eta(s)\eta^T(-s) = \Psi$ for some constant Ψ. The parameterization matrix $\varphi(s)$ is found by solving for $\varphi(s)$ in the equation

$$\eta(s) = \varphi(s)P(s) \qquad (8)$$

which has only one unique solution $\varphi(s) = \eta(s)P^{-1}(s)$. ∎

Now, we are ready to present the main theorem on design of innovation filters.

Theorem 3. *If $Z(s)$ is full rank, an innovation filter exists if and only if*

$$\forall i. \textit{row-deg}_i N_{M_s}(s)\begin{pmatrix} D_n \\ B_n \end{pmatrix} = \textit{row-deg}_i N_{M_s}(s)$$

and $Z(s)$ has no roots on the imaginary axis. Furthermore, if an innovation filter exist, all innovation filters can be parameterized as

$$Q(s) = \eta(s)P^{-1}(s)N_{M_s}(s)P_x$$

where $P(s)$ is a spectral co-factor of $Z(s)$ and $\eta(s)$ is any strictly stable, full-rank matrix, such that $\eta(s)\eta^T(-s)$ is constant.

Proof. According to Lemma 1 and Lemma 2, an innovation filter exists if and only if there exists an $\eta(s)$ such that

$$Q(s) = \eta(s)P^{-1}(s)N_{M_s}(s)P_x$$

is stable, proper, $\eta(s)\eta^T(-s)$ is constant and full-rank of dimension $r \times r$ with $r = \dim\ \mathcal{N}_L(M_s(s))$.

First, assume $Q(s)$ is an innovation filter and that $Z(s)$ has a zero at $s_0 = j\omega_0$. Since $Q(s)$ is strictly stable, $\lim_{s\to j\omega_0} Q(s)$ exists. But, $Z(s)$ has a zero at s_0 implies that $P(s_0)$ is rank deficient. Since, according to assumption, $Q(s_0)$ exists, it must hold that $\eta(s)$ looses rank at s_0 since $N_{M_s}(s)P_x$ is irreducible. However, this contradicts $\Psi = \eta(s)\eta^T(-s)$ being full-rank which gives that full-rank of $Z(s)$ on the imaginary axis is a necessary condition for $Q(s)$ to be stable.

Now, assume $Q(s)$ is an innovation filter and that there exists an i such that

$$\text{row-deg}_i\ N_{M_s}(s)\begin{pmatrix} D_n \\ B_n \end{pmatrix} < \text{row-deg}_i\ N_{M_s}(s) \qquad (9)$$

Partition $N_{M_s}(s) = [V_1(s)\ V_2(s)]$ according to the block-structure of (3). It is possible to show[1] that $V_1(s)$ is row-reduced, $\text{row-deg}_i N_{M_s}(s) = \text{row-deg}_i V_1(s)$, and that $V_2(s) = V_1(s)C(sI-A)^{-1}$. Since $V_1(s)$ is row-reduced, we can rewrite (9) as

$$\text{row-deg}_i\ S_{V_1}(s)V_{1,hr}D_n + \tilde{V}_1(s)D_n + V_2(s)B_n < \text{row-deg}_i\ V_1(s)$$

where $V_{1,hr}$ is the high-degree-coefficient matrix of $V_1(s)$ and $S_{V_1}(s)$ is a diagonal matrix with s_i^{μ} in the diagonal and μ_i is the i:th row-degree of $V_1(s)$.

Since the row-degrees of $\tilde{V}_1(s)$ and $V_2(s)$ is strictly less than the row-degrees of $S_{V_1}(s)$, the inequality can only be fulfilled if $V_{1,hr}D_n$ does not have full row-rank. This also gives that

$$\lim_{s\to\infty} V_1(s)D_n = \lim_{s\to\infty} S_{V_1}(s)V_{1,hr}D_n \qquad (10)$$

does not have full row-rank. Now, since $Q(s)$ is an innovation filter, there exist an $\eta(s)$ such that $Q(s) = \eta(s)P^{-1}(s)N_{M_s}(s)P_x$ and $H(s)H^T(-s)$ is square, full-rank, and constant where

$$H(s) = \eta(s)P^{-1}(s)N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}$$

[1] See (Frisk and Nyberg, 2001) for proofs

But, when s goes to infinity, it holds that

$$\lim_{s\to\infty} H(s) = \\ = \lim_{s\to\infty} \eta(s)P^{-1}(s)V_1(s)\left(C(sI-A)^{-1}B_n + D_n\right) = \\ = \lim_{s\to\infty} \eta(s)P^{-1}(s)V_1(s)D_n$$

which does not have full row-rank due to (10) and the fact that $\eta(s)$ and $P(s)$ is square and full-rank. Thus, $\lim_{s\to\infty} H(s)$ does not have full rank which contradicts that $H(s)H^T(-s)$ is constant and full-rank.

Now it has been proven that the two conditions in the theorem, $Z(s)$ full-rank and the row-degree condition, is necessary conditions for the existence of an innovation filter. Next, sufficiency will be proven. Since $Z(s)$ does not have zeros on the imaginary axis, a spectral co-factor $P(s)$ will be strictly stable and row-reduced with row-degrees satisfying

$$\text{row-deg}_i\ P(s) = \text{row-deg}_i\ N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}$$

See (Kwakernaak and Šebek, 1994) for proofs of these claims. Then, (Kailath, 1980, Theorem 6.3-12) gives that

$$Q(s) = P^{-1}(s)N_{M_s}(s)P_x$$

will be proper, strictly stable and fulfill all requirements in Definition 1, i.e. $Q(s)$ is an innovation filter.

Finally, if $Q(s)$ is an innovation filter, it is immediate that $Q'(s)$ is an innovation filter if and only if $Q'(s) = \eta(s)Q(s)$ where $\eta(s)$ is a square, full-rank, all-pass link i.e. $\eta(s)\eta^T(-s)$ is constant and full rank. ■

Matlab-code for innovation filter design

To illustrate the simplicity of the design algorithm, a complete Matlab-session (requires control and polynomial toolbox) for design of an innovation filter is given by:

```
Ms = [C Dd;-(s*eye(nx)-A) Bd];
Px = [eye(m) -Du;zeros(nx,m) -Bu];
Nms = null(Ms.').';
Z = Nms*[Dn;Bn]*[Dn;Bn]'*Nms';
[P,J] = spf(Z.'); P = P.';
[Qa,Qb,Qc,Qd] = lmf2ss(Nms*Px,P);
Q = ss(Qa,Qb,Qc,Qd);
```

As shown above, no diagnosis specific code need to be developed and the design procedure solely relies on high performance numerical routines in established Matlab toolboxes. The numerical performance in diagnosis applications of the above code is illustrated in (Frisk, 2001).

Descriptor systems

An extension of the above design algorithm for descriptor systems

$$E\dot{x} = Ax + B_u u + B_d d + B_n n + B_f f \\ y = Cx + D_u u + D_d d + D_n n + D_f f$$

where E is non-singular or non-square is immediate by letting

$$M_s(s) = \begin{bmatrix} C & D_d \\ -(sE-A) & B_d \end{bmatrix}$$

instead of (3). Design of innovation filters for descriptor examples is thoroughly described in (Frisk, 2001).

Relations to solution in (Nikoukhah, 1994)

Since the solution provided by Theorem 3 solves the problem posed in (Nikoukhah, 1994), equivalent (but not identical) results can be found in Nikhoukhah's paper. The main differences between the solutions is that the algorithm provided here has been generalized to solve the more general whitening residual generator design problem and also applies to descriptor systems.

4.1 Design of whitening residual generators

The design procedure for whitening residual generators is a bit more complex due to the increased design freedom, resulting in a possibly more involved design procedure; especially when $Z(s)$ has zeros on the imaginary axis and/or if no row of $P^{-1}(s)N_{M_s}(s)P_x$ is proper.

Due to space constraints, the interested reader is referred to (Frisk, 2001) for details, here only a sufficient condition for the existence of a whitening residual generator is included. This result is however enough to illustrate whitening residual generators in the examples in Section 5.

Theorem 4. *If $Z(s)$ is full rank with no zeros on the imaginary axis, a whitening residual generator exists if*

$$\exists i.\textit{row-deg}_i N_{M_s}(s)\begin{pmatrix} D_n \\ B_n \end{pmatrix} = \textit{row-deg}_i N_{M_s}(s)$$

Proof. Since, according to assumption in the theorem, $Z(s)$ has no zeros on the imaginary axis, strict stability of the residual generator is assured. By (Kwakernaak and Šebek, 1994), $P(s)$ is row-reduced and the row-degrees of $P(s)$ equals the row-degrees of $N_{M_s}(s)\begin{pmatrix} D_n \\ B_n \end{pmatrix}$. Then (Kailath, 1980, Theorem 6.3-12) assures the existence of a whitening residual generator. ■

5 Design examples

This section includes 3 design examples that illustrates different aspects of the design problem and the proposed design algorithm. The examples are based around a linearized airplane model which has been used previously in e.g. (Frisk and Nyberg, 1999) to demonstrate the deterministic design problem. In the first example, a complete design of an innovation filter and a whitening residual generator is shown. In the second example the noise environment is changed and it is shown that no innovation filter or whitening residual generator exists. In the third example, using a third noise setup, it is shown that an innovation filter does not exists but a whitening residual generator exists that has acceptable fault sensitivity.

All calculations is done in Matlab using Polynomial Toolbox 2.5 for Matlab 5 (2001). All functions used is included in the toolbox and no diagnosis specific code is needed.

5.1 Design Example: Aircraft Dynamics

The model used in these examples is taken from (Maciejowski, 1989) and represents a linearized model of vertical-plane dynamics of an aircraft. The model has 5 states, 3 inputs, and 3 outputs. The nominal model is given in state-space form and parameter value can be found in (Maciejowski, 1989).

Here, assume additive sensor-faults (denoted f_1, f_2, and f_3), and additive actuator-faults (denoted f_4, f_5, and f_6). Also, assume that the process is influenced by additive white noise, both in the dynamic and measurement equations. The model can now easily be written on the form (1). The noise is assumed white with unit covariance.

The design goal in all the three examples based on this model are a residual generator $Q(s)$ that decouples faults in the elevator angle actuator, i.e. f_6, and produces a white residual in the fault-free case. The difference in the designs are different noise assumptions. The motive for the decoupling of f_6 is fault isolation by structured residuals (Gertler, 1991).

Process and measurement noise

In this first example, both measurement noise and process noise is considered and state-space matrices B_n and D_n is set to

$$B_n = [I_5 \ \ 0_{5\times 3}] \qquad D_n = [0_{3\times 5} \ \ I_3]$$

First, an innovation filter design is performed. Calculations in MATLAB give

$$N_{M_s}(s) = \begin{bmatrix} -0.07s & -s-0.054 & \cdots \\ 0.99s^2+0.64s & -0.07s^2-0.049s-0.3 & \cdots \end{bmatrix.$$
$$\begin{matrix} \cdots & -0.091 & -0.07 & \cdots \\ \cdots & 0.044s^2-0.048s-0.73 & 0.99s+0.64 & \cdots \end{matrix}$$
$$\left.\begin{matrix} \cdots & -1 & 0 & 0 & 0 \\ \cdots & -0.07s-0.045 & 0.044s+1.1 & 0.044 & -1 \end{matrix}\right] \quad (11)$$

Thus, the dimension of the null-space $\mathcal{N}_L(M_s(s))$ is 2, i.e. there exists exactly two linearly independent numerators that decouples f_6.

Then, matrix $Z(s)$ is computed and it is easy to verify that it is full-rank, i.e. it is not possible to perfectly decouple the noise. A J-spectral co-factorization of $Z(s)$ gives the spectral factor:

$$P(s) = \begin{bmatrix} -0.99s-1 & -0.12s-0.084 \\ 0.12s^2+0.32s+0.068 & -0.99s^2-2.2s-1.8 \end{bmatrix}$$

The spectral factor $P(s)$ is strictly stable which can be seen by computing the zeros of the invariant polynomials. Matlab gives the zeros $s = -1.0196$ and $s = -1.1124 \pm j0.7305$ which lies in the open left-half plane.

Checking for existence of innovation filter according to Theorem 3 gives:

$$\text{row-deg } N_{M_s}(s) = \{1,2\}$$
$$\text{row-deg } N_{M_s}(s) \begin{bmatrix} D_n \\ B_n \end{bmatrix} = \{1,2\}$$

i.e. an innovation filter exists and can be formed as $Q(s) = P^{-1}(s)N_{M_s}(s)P_x$.

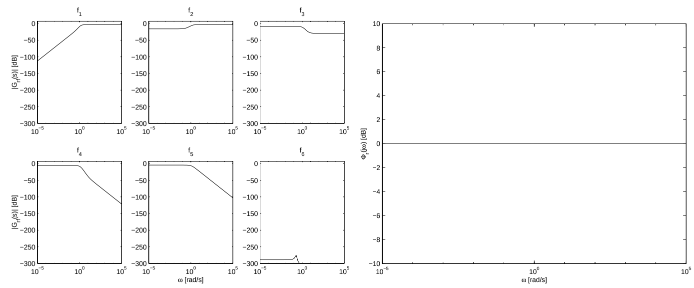
Next, a scalar whitening residual generator is to be designed. A whitening residual generator can be formed as

$$Q(s) = \eta(s)P^{-1}(s)N_{M_s}(s)P_x$$

where $\eta(s)\eta^T(-s)$ is constant. With

$$\eta(s) = \frac{1}{\sqrt{2}}[1 \ 1]$$

a 3:rd order realizable and strictly stable residual generator is obtained. The order of the residual generator is, due to the choice of $\eta(s)$, equal to the sum of row-degrees of $P(s)$. Figure 1 shows how the faults influence the residual and that the fault-free spectrum $\Phi_r(j\omega)$ is 1 for all ω as expected. Especially note that the desired decoupling of fault f_6 has succeeded while keeping the spectrum of r constant for all ω. Note that the DC gain from fault f_1 to the residual is zero which of course is bad for detectability, however it can easily be shown that there exists no strictly stable linear residual generator with non-zero DC-gain.



(a) Magnitude bode plots for the faults to the residual

(b) Spectrum $\Phi_r(j\omega)$

Figure 1: Result of the first design

Only process noise

In this second example, only process noise is considered and state-space matrices B_n and D_n is set to:

$$B_n = I_5 \qquad D_n = 0_{3\times 5}$$

The null-space basis $N_{M_s}(s)$ is identical to the first example (11). The row-degrees of $N_{M_s}(s)$ is $\{1,2\}$ and the row-degrees of $N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}$ is $\{0,1\}$, i.e. no innovation filter exists according to Theorem 3. In (Frisk, 2001) it is also shown that no whitening residual generator exists either.

Noise on all states and sensor 3

In this final case the process is subjected to noise on all states and on sensor 3, i.e. the matrices B_n and D_n are given by

$$B_n = [I_5 \ 0_{5\times 1}] \qquad D_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

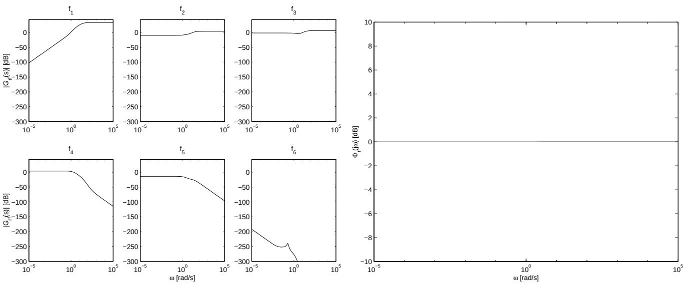
Now, computing $N_{M_s}(s)$ and $Z(s)$ as before gives that $Z(s)$ is strictly stable and that the row-degrees of $N_{M_s}(s)$ is (as before) $\{1,2\}$ and row-degrees of $N_{M_s}(s)\begin{bmatrix} D_n \\ B_n \end{bmatrix}$ is $\{0,2\}$. This gives that no innovation filter exists according to Theorem 3. But because $Z(s)$ has no imaginary zeros and the row-degree of the second row of $N_{M_s}(s)$ does not decrease when multiplied by the noise distribution matrices, Theorem 4 proves existence of a whitening residual generator. A whitening residual generator can be formed by

$$Q(s) = \eta(s)P^{-1}(s)N_{M_s}(s)P_x$$

which is proper when selecting

$$\eta(s) = \begin{bmatrix} -0.0994 & 0.995 \end{bmatrix}$$

Figures 2 shows the fault influence on the residual and the spectrum of the fault-free residual. Here it is clear that the residual still is able to detect all faults, besides from fault f_6 which should be decoupled according to design specifications. Thus, even though an innovation filter didn't exist, a whitening residual generator with satisfactory fault detectability properties existed. This residual generator could not have been designed algorithm proposed in (Nikoukhah, 1994).



(a) Magnitude bode plots for the faults to the residual

(b) Spectrum $\Phi_r(j\omega)$

Figure 2: Result of the second design

It is straightforward to realize that innovation filters preserve any fault detectability properties since $\varphi(s)$ is invertible, i.e. the number of outputs of an innovation filter equals the dimension of $\mathcal{N}_L(M_s(s))$. However, if an innovation filter does not exist, there may very well exist a whitening residual generator with desirable fault detectability properties which was the case in this example.

6 Discrete-time systems

In the deterministic case, time-discrete systems and time-continuous systems could be handled analogously by replacing s with z and proper with causal (Frisk and Nyberg, 2001). In the stochastic case, small but important differences exists. The main difference between the time-continuous and time-discrete cases is that properness/causality of the residual generator can always be achieved by e.g. inserting a number of time-delays in the residual generator. Thus, it is immediate to prove that the existence conditions for full-rank innovation filters and whitening residual generators is identical to the time-continuous case where the properness condition has been removed, thus, a time-discrete version of Theorem 3 becomes:

Theorem 5. *If $Z(z)$ is full rank, an innovation filter exists if and only if $Z(z)$ has no roots on the imaginary axis.*

When existence has been ensured, the design procedure is identical to the time-continuous case (using discrete-time spectral factorization algorithms).

7 Conclusions

A polynomial design algorithm for linear residual generation for stochastic state-space and descriptor systems in both continuous and discrete time has been considered. The problem formulated is based on innovation filters formulated by Nikoukhah (1994). The problem formulation is further developed to a new class of residual generators, whitening residual generators.

The two main steps in the design algorithm is computation of a polynomial basis for the left null-space of a polynomial matrix followed by a J-spectral co-factorization of a para-hermitian polynomial matrix. For both these operations, good numerical tools exists and the algorithm is successfully demonstrated on a number of non-trivial examples.

References

M. Basseville and I.V. Nikiforov. *Detection of Abrupt Changes - Theory and Application*. Prentice Hall, 1993.

J. Chen and R.J. Patton. A re-examination of fault detectability and isolability in linear dynamic systems. Fault Detection, Supervision and Safety for Technical Processes, pages 567–573, Espoo, Finland, 1994. IFAC.

E.Y. Chow and A.S. Willsky. Analytical redundancy and the design of robust failure detection systems. *IEEE Trans. on Automatic Control*, 29(7):603–614, 1984.

E. Frisk. *Residual Generation for Fault Diagnosis*. PhD thesis, Linköping University, 2001.

E. Frisk and M. Nyberg. Using minimal polynomial bases for fault diagnosis. European Control Conference, 1999.

E. Frisk and M. Nyberg. A minimal polynomial basis solution to residual generation for fault diagnosis in linear systems. *Automatica*, 37(9):1417–1424, 2001.

J. Gertler. Analytical redundancy methods in fault detection and isolation; survey and synthesis. IFAC Fault Detection, Supervision and Safety for Technical Processes, pages 9–21, Baden-Baden, Germany, 1991.

T. Kailath. *Linear Systems*. Prentice Hall, 1980.

H. Kwakernaak. A descriptor algorithm for the spectral factorization of polynomial matrices. In *Proc. Robust Control Design*, Prague, Czech Republic, 2000. IFAC.

H. Kwakernaak and M. Šebek. Polynomial J-spectral factorization. *IEEE Tran. on Automatic Control*, 39(2):315–328, 1994.

J.M. Maciejowski. *Multivariable Feedback Design*. Addison Wesley, 1989.

R. Nikoukhah. Innovations generation in the presence of unknown inputs: Application to robust failure detection. *Automatica*, 30(12):1851–1867, 1994.

The Polynomial Toolbox 2.5 for Matlab 5. Polyx, Czech Republic. URL: http://www.polyx.com, 2001.